UNIFORM DISTRIBUTION OF ORBITS OF LATTICES ON SPACES OF FRAMES

ALEXANDER GORODNIK

ABSTRACT. We study distribution of orbits of a lattice $\Gamma \subseteq \mathrm{SL}(n, \mathbb{R})$ in the the space $\mathcal{V}_{n,l}$ of l-frames in $\mathbb{R}^n$ ($1 \leq l \leq n-1$). Examples of dense Γ-orbits are known from the work of Dani, Raghavan, and Veech. We show that dense orbits of Γ are uniformly distributed in $\mathcal{V}_{n,l}$ with respect to an explicitly described measure. We also establish analogous result for lattices in $\mathrm{Sp}(n, \mathbb{R})$ that act on the space of isotropic n-frames.

1. INTRODUCTION

Let $G = \mathrm{SL}(n, \mathbb{R})$ and $\mathcal{V}_{n,l}$ be the space of l-frames in $\mathbb{R}^n$ (i.e. the space of l-tuples of linearly independent vectors in $\mathbb{R}^n$), $1 \leq l \leq n$. The group G acts on this space as follows:

$$g \cdot (v_1, \ldots, v_l) = (gv_1, \ldots, gv_l), \quad g \in \mathrm{SL}(n, \mathbb{R}), \ (v_1, \ldots, v_l) \in \mathcal{V}_{n,l}.$$

The action is transitive for $l < n$. Let Γ be a lattice in G; that is, a discrete subgroup in G such that the factor space $\Gamma \backslash G$ has finite volume (e.g. $\Gamma = \mathrm{SL}(n, \mathbb{Z})$). The main result of this paper concerns distribution of Γ-orbits in $\mathcal{V}_{n,l}$.

When $l = n$, every orbit of Γ is discrete. The situation becomes much more interesting for $l < n$. Let us recall known results:

Theorem 1. (Dani, Raghavan [DR80]**)** *Let* $\Gamma = SL(n, \mathbb{Z})$, *and* $v = (v_1, \ldots, v_l)$ *be an l-frame in $\mathbb{R}^n$, $1 \leq l \leq n-1$. Then the orbit $\Gamma \cdot v$ is dense in $\mathcal{V}_{n,l}$ iff the space spanned by $\{v_i : i = 1, \ldots, l\}$ contains no nonzero rational vectors.*

Theorem 2. (Veech [Ve77]**)** *If Γ is a cocompact lattice in G, then every orbit of Γ in $\mathcal{V}_{n,l}$, $1 \leq l \leq n-1$, is dense.*

Theorems 1 and 2 provide examples of dense Γ-orbits in $\mathcal{V}_{n,l}$. Here we show that dense Γ-orbits are uniformly distributed with respect to an explicitly described measure on $\mathcal{V}_{n,l}$. This measure is $\frac{dv}{\mathrm{Vol}(v)}$, where dv is the Lebesgue measure on $\prod_{i=1}^{l} \mathbb{R}^n$, and $\mathrm{Vol}(v)$ is the l-dimensional volume of the frame v.

Note that the measure dv is G-invariant, and it is unique up to a constant. However, orbits of Γ are equidistributed with respect to the measure $\frac{dv}{\mathrm{Vol}(v)}$, which is not G-invariant. This phenomenon was already observed by Ledrappier [Le99].

Define a norm on $M(n, \mathbb{R})$ by

$$\|x\| = \left(\mathrm{Tr}(^txx)\right)^{1/2} = \left(\sum_{i,j} x_{ij}^2\right)^{1/2} \quad \text{for } x = (x_{ij}) \in M(n, \mathbb{R}). \tag{1}$$

For $T > 0$, $\Omega \subseteq \mathcal{V}_{n,l}$, $v^0 \in \mathcal{V}_{n,l}$, put

$$N_T(\Omega, v^0) = |\{\gamma \in \Gamma : \|\gamma\| < T, \gamma \cdot v^0 \in \Omega\}|. \tag{2}$$

We determine asymptotic behavior of $N_T(\Omega, v^0)$ as $T \to \infty$. This result gives a quantitative strengthening of Theorems 1 and 2, and it can be interpreted as uniform distribution of dense orbits of Γ in $\mathcal{V}_{n,l}$.

Theorem 3. *Let Γ be a lattice in $SL(n, \mathbb{R})$. Let $v^0 \in \mathcal{V}_{n,l}$ be an l-frame in $\mathbb{R}^n$ such that $\Gamma \cdot v^0$ is dense in $\mathcal{V}_{n,l}$. Let Ω be a relatively compact Borel subset of $\mathcal{V}_{n,l}$ such that $\int_{\partial \Omega} dv = 0$. Then*

$$N_T(\Omega, v^0) \sim a_{n,l} \frac{\text{Vol}(v^0)^{1-n}}{\bar{\mu}(\Gamma \backslash G)} \left(\int_\Omega \frac{dv}{\text{Vol}(v)} \right) T^{(n-1)(n-l)} \quad as \quad T \to \infty, \tag{3}$$

where $a_{n,l}$ is a constant (which is computed in (45) below), and $\bar{\mu}$ is a G-invariant measure on $\Gamma \backslash G$ (which is defined in (29) below).

Remark. The term $T^{(n-1)(n-l)}$ in (3) comes from the asymptotics of the volume of the set $\{h \in H : \|h\| < T\}$ in the stabilizer H of v^0 with respect to the measure on H which is determined by the choice of the Haar measures on G and $\mathcal{V}_{n,l} = G \cdot v^0$ (see Section 2).

For $n = 2$ and $l = 1$, this theorem was proved by Ledrappier [Le99] for general Γ and by Nogueira [No00] for $\Gamma = SL(2, \mathbb{Z})$ and max-norm using different methods.

Combining Theorems 1 and 3, we get:

Corollary 4. *Let $\Gamma = SL(n, \mathbb{Z})$. Let $v^0 = (v_1^0, \ldots, v_l^0) \in \mathcal{V}_{n,l}$ be an l-frame in $\mathbb{R}^n$ such that the space $\langle v_1^0, \ldots, v_l^0 \rangle$ contains no nonzero rational vectors. Let Ω be a relatively compact Borel subset of $\mathcal{V}_{n,l}$ such that $\int_{\partial \Omega} dv = 0$. Then*

$$N_T(\Omega, v^0) \sim b_{n,l} \text{Vol}(v^0)^{1-n} \left(\int_\Omega \frac{dv}{\text{Vol}(v)} \right) T^{(n-1)(n-l)} \quad as \quad T \to \infty, \tag{4}$$

where $b_{n,l}$ is a constant (which is computed in (83) below).

Example Figure 1 shows a part of the the orbit $SL(2, \mathbb{Z})v^0$ for $v^0 = {}^t(\sqrt{2}, \sqrt{3})$. By the result of Ledrappier, this orbit is uniformly distributed in $\mathbb{R}^2$ with respect to the measure $\frac{dxdy}{\sqrt{x^2+y^2}}$.

Dani and Raghavan also considered orbits of frames under $\text{Sp}(n, \mathbb{Z})$. Denote

$$J = \left(\begin{array}{c|c} 0 & E \\ \hline -E & 0 \end{array} \right),$$

where E is the identity $n \times n$ matrix. The symplectic form $(x, y) \mapsto {}^t x J y$ will be denoted by J too. Let

$$G = \text{Sp}(n, \mathbb{R}) = \{g \in SL(2n, \mathbb{R}) : {}^t g J g = J\}$$

and $\Gamma = \text{Sp}(n, \mathbb{Z})$. A frame $(v_1, \ldots, v_s)$ is called *isotropic* if the symplectic form J is 0 on the space spanned by $\{v_i : i = 1, \ldots, s\}$.

Theorem 5. (Dani, Raghavan [DR80]) *Let $v = (v_1, \ldots, v_n)$ be an isotropic frame in $\mathbb{R}^{2n}$. Then $\Gamma \cdot v$ is dense in the space of isotropic n-frames iff the space spanned by $\{v_i : i = 1, \ldots, n\}$ contains no nonzero rational vectors.*
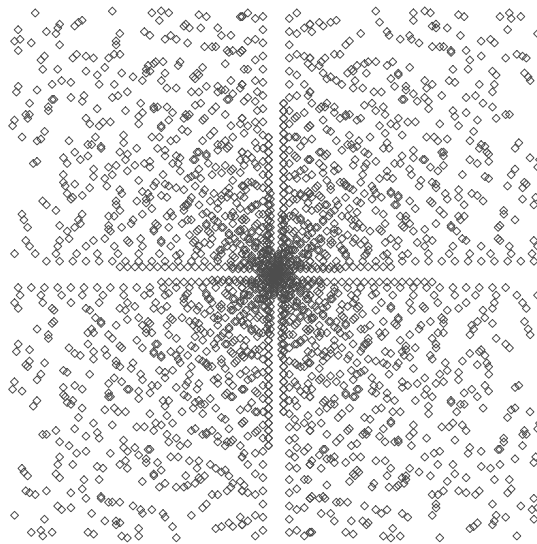
FIGURE 1.

A result similar to Theorem 3 holds in this case too. Denote by $\mathcal{W}_n$ the space of $2n$-dimensional n-frames that are isotropic with respect to the standard symplectic form J. Note that $\mathcal{W}_n$ is an open subset of an algebraic set in $\prod_{i=1}^{n} \mathbb{R}^{2n}$. Since by Witt's theorem $\mathrm{Sp}(n, \mathbb{R})$ acts transitively on $\mathcal{W}_n$, $\mathcal{W}_n$ is a submanifold of $\prod_{i=1}^{n} \mathbb{R}^{2n}$.

We improve Theorem 5 by showing that dense orbits of Γ are uniformly distributed:

Theorem 6. *Let Γ be a lattice in $Sp(n, \mathbb{R})$, and $v^0 \in \mathcal{W}_n$ be such that $\Gamma \cdot v^0$ is dense in $\mathcal{W}_n$. Let Ω be a relatively compact Borel subset of $\mathcal{W}_n$ such that the boundary of Ω has measure 0 in the Lebesgue measure class. Then*

$$N_T(\Omega, v^0) \sim \lambda_{v^0}(\Omega) T^{n(n+1)/2} \quad as \quad T \to \infty$$

for some measure λ_{v^0} on $\mathcal{W}_n$ in the Lebesgue measure class, which can be explicitly computed.

Note that the measure λ_{v^0} is not $\mathrm{Sp}(n, \mathbb{R})$-invariant.

In the next section we show how to derive asymptotic distribution for counting functions similar to $N_T(\Omega, v^0)$ from uniform distribution of orbits of subgroups of G in the space $\Gamma \backslash G$. In section 3, we consider the case $G = \mathrm{SL}(n, \mathbb{R})$. First, for convenience of the reader, we sketch an easy proof of Theorems 1 and 2 based on topological rigidity of unipotent flows, which was established by Ratner [Rat91b]. Then we introduce a decomposition of G based on the Iwasawa decomposition, and obtain results on volume of balls in the subgroup B_l^o, which is defined below. This allows us to use results from Section 2 to prove Theorem 3 and Corollary 4 modulo ergodic theorem along balls in the group B_l^o (Theorem 20). In Section 4, we prove the ergodic theorem for B_l^o. Note that for $l = n - 1$ it is a special case of the result of Shah [Sh94]. The proof of the ergodic theorem is similar to the proof of Theorem 2 from [Go02]. Finally, in Section 5 we consider the case $G = \mathrm{Sp}(n, \mathbb{R})$, and prove Theorem 6. The method of the proof is similar to one used for Theorem 3: we use Iwasawa decomposition for $\mathrm{Sp}(n, \mathbb{R})$ and uniform distribution of large unipotent subgroups due to Shah [Sh94]. In the Appendix, we prove some technical volume estimates and Corollary 4.

Remark. In the definition of $N_T(\Omega, v^0)$, we used the norm (1). The fact that this norm is invariant under the orthogonal group made our calculations easier. However, one can prove similar results for every norm on $\mathrm{M}(n, \mathbb{R})$ with possibly different limit measure in the Lebesgue measure class.

2. SOME LIMIT THEOREMS

In this section we establish asymptotics of some counting functions.

Let $G \subseteq \mathrm{SL}(n, \mathbb{R})$ be a Lie group, Γ a lattice in G, and H a Lie subgroup of G. Denote by ϱ a right Haar measure on H. Let μ be a Haar measure on G, and $\bar{\mu}$ be the measure on $\Gamma\backslash G$ such that

$$\int_G f d\mu = \int_{\Gamma\backslash G} \left(\sum_{\gamma \in \Gamma} f(\gamma g) \right) d\bar{\mu}(g), \quad f \in C_c(G).$$

Throughout this section, we assume that for some $M > 0$ and every $c > 0$, $m \in \mathbb{R}$,

$$\lim_{T \to \infty} \frac{\varrho(H_{cT+m})}{\varrho(H_T)} = c^M. \tag{5}$$

where $H_T = \{h \in H : \|h\| < T\}$, and for every $x \in \Gamma\backslash G$ such that xH is dense in $\Gamma\backslash G$ and $\tilde{f} \in C_c(\Gamma\backslash G)$,

$$\frac{1}{\varrho(H_T)} \int_{H_T} \tilde{f}(xh^{-1}) d\varrho(h) \to \frac{1}{\bar{\mu}(\Gamma\backslash G)} \int_{\Gamma\backslash G} \tilde{f} d\mu \quad \text{as} \quad T \to \infty. \tag{6}$$

First, we prove an elementary lemma:

Lemma 7. *Let $(V, \|\cdot\|)$ be a normed vector space, G be a topological group, and $\rho : G \to \mathcal{B}(V)^*$ (= the space of bounded invertible linear operators on V) be a continuous map (w.r.t. norm topology). Then for any $g_0 \in G$ and $k > 1$, there exists a neighborhood $\mathcal{O}_{g_0}$ of g_0 in G such that for any $g \in \mathcal{O}_{g_0}$ and $v \in V$,*

$$k^{-1}\|\rho(g_0)v\| \le \|\rho(g)v\| \le k\|\rho(g_0)v\|.$$

Proof. Take

$$\mathcal{O}_{g_0} = \{g : \|\rho(g_0)\rho(g)^{-1}\| < k, \|\rho(g)\rho(g_0)^{-1}\| < k\}.$$

Then for $g \in \mathcal{O}_{g_0}$,

$$\|\rho(g_0)v\| \le \|\rho(g_0)\rho(g)^{-1}\| \cdot \|\rho(g)v\| < k\|\rho(g)v\|.$$

Similarly, for $g \in \mathcal{O}_{g_0}$,

$$\|\rho(g)v\| \le \|\rho(g)\rho(g_0)^{-1}\| \cdot \|\rho(g_0)v\| < k\|\rho(g_0)v\|.$$

$\square$

For $g \in \mathrm{SL}(n, \mathbb{R})$, denote by $\hat{g} : x \mapsto gxg^{-1}$ the inner automorphism corresponding to g. For a subgroup L of G, denote by $N_G(L)$ the normalizer of L in G. For $g \in N_G(H)$, define

$$\Delta_H(g) = \left| \det \left(\mathrm{Ad}(g)|_{\mathrm{Lie}(H)} \right) \right| \tag{7}$$

where $\mathrm{Ad}(g)$ is the adjoint transformation of the Lie algebra of G.

Proposition 8. *Let $x_0 \in G$ be such that $\Gamma x_0 H$ is dense in $\Gamma \backslash G$. Let*

$$
\begin{aligned}
g \mapsto c_g : & \quad G \to N_G(H), \\
g \mapsto d_g : & \quad G \to H, \\
g \mapsto e_g : & \quad G \to \mathbb{R}, \\
g \mapsto m_g : & \quad G \to \mathbb{R}_+,
\end{aligned}
$$

be continuous maps that factor through $G/\hat{x}_0(H)$. Then for every $f \in C_c(G)$,

$$\lim_{T \to \infty} \frac{1}{\varrho(H_T)} \sum_{\gamma \in \Gamma} \int_{m_\gamma^2 \|\hat{c}_\gamma(hd_\gamma)\|^2 + e_\gamma < T^2} f(\gamma x_0 h^{-1} x_0^{-1}) d\varrho(h) \tag{8}$$

$$= \frac{1}{\bar{\mu}(\Gamma \backslash G)} \int_G \frac{f(g)}{m_g^M \cdot \Delta_H(c_g)} d\mu(g).$$

Proof. We shall assume without loss of generality that $f \geq 0$.

There exist real M_1 and M_2 such that $M_1 \leq e_g \leq M_2$ for $g \in \mathrm{supp}(f)\hat{x}_0(H)$. Then for $g \in \mathrm{supp}(f)\hat{x}_0(H)$,

$$\{h : m_g^2 \|\hat{c}_g(hd_g)\|^2 + e_g < T^2\} \subseteq \{h : m_g \|\hat{c}_g(hd_g)\| < (T^2 - M_1)^{1/2}\}. \tag{9}$$

Denote $\tilde{f}(\Gamma g) = \sum_{\gamma \in \Gamma} f(\gamma g)$. Note that $\tilde{f} \in C_c(\Gamma \backslash G)$.

Let $r > 1$ and $\varepsilon > 0$. By Lemma 7, for any $g_0 \in G$ there exists a neighborhood $\mathcal{O}_{g_0}$ of g_0 such that

$$r^{-1} \|\hat{c}_{g_0}(v)\| < \|\hat{c}_g(v)\| < r \|\hat{c}_{g_0}(v)\|, \tag{10}$$

$$r^{-1} \|v\hat{c}_{g_0}(d_{g_0})\| < \|v\hat{c}_{g_0}(d_g)\| < r \|v\hat{c}_{g_0}(d_{g_0})\| \tag{11}$$

for all $g \in \mathcal{O}_{g_0} \hat{x}_0(H)$ and $v \in M(n, \mathbb{R})$. Moreover, $\mathcal{O}_{g_0}$ can be taken such that

$$\left| \frac{1}{m_g^M \cdot \Delta_H(c_g)} - \frac{1}{m_{g_0}^M \cdot \Delta_H(c_{g_0})} \right| < \varepsilon,$$

and

$$r^{-1} m_{g_0} \leq m_g \leq r m_{g_0}$$

for all $g \in \mathcal{O}_{g_0} \hat{x}_0(H)$.

Note that for every $v \in N_G(H)$, $\Gamma x_0 v H$ is dense in $\Gamma \backslash G$. Therefore, by (6), for every $u \in G$ and $v \in N_G(H)$,

$$
\begin{aligned}
\lim_{T \to \infty} \frac{1}{\varrho(H_T)} \int_{H_T} \tilde{f}(\Gamma x_0 v h^{-1} u) d\varrho(h) & = \frac{1}{\bar{\mu}(\Gamma \backslash G)} \int_{\Gamma \backslash G} \tilde{f}(yu) d\bar{\mu}(y) \\
& = \frac{1}{\bar{\mu}(\Gamma \backslash G)} \int_{\Gamma \backslash G} \tilde{f} d\bar{\mu}. \tag{12}
\end{aligned}
$$

To prove (8), we first suppose that $\mathrm{supp}(f) \subseteq \mathcal{O}_{g_0}$ for some $g_0 \in G$. Put $c_0 = c_{g_0}$, $d_0 = d_{g_0}$, $m_0 = m_{g_0}$. Then using (9), (10), and (11), we get

$$\sum_{\gamma \in \Gamma} \int_{m_\gamma^2 \|\hat{c}_\gamma(hd_\gamma)\|^2 + e_\gamma < T^2} f(\gamma x_0 h^{-1} x_0^{-1}) d\varrho(h)$$

$$\leq \sum_{\gamma \in \Gamma} \int_{m_\gamma \|\hat{c}_\gamma(hd_\gamma)\| < (T^2 - M_1)^{1/2}} f(\gamma x_0 h^{-1} x_0^{-1}) d\varrho(h)$$

$$\leq \sum_{\gamma \in \Gamma} \int_{r^{-2} m_0 \|\hat{c}_0(h)\hat{c}_0(d_\gamma)\| < (T^2 - M_1)^{1/2}} f(\gamma x_0 h^{-1} x_0^{-1}) d\varrho(h)$$

$$\leq \sum_{\gamma \in \Gamma} \int_{r^{-3} m_0 \|\hat{c}_0(h)\hat{c}_0(d_0)\| < (T^2 - M_1)^{1/2}} f(\gamma x_0 h^{-1} x_0^{-1}) d\varrho(h)$$

$$= \int_{\|h\| < r^3 m_0^{-1}(T^2 - M_1)^{1/2}} \tilde{f}(\Gamma x_0 c_0^{-1} \hat{c}_0(d_0) h^{-1} c_0 x_0^{-1}) \frac{d\varrho(h)}{\Delta_H(c_0)}.$$

Thus, by (12) and (5),

$$\limsup_{T \to \infty} \frac{1}{\varrho(H_T)} \sum_{\gamma \in \Gamma} \int_{m_\gamma^2 \|\hat{c}_\gamma(hd_\gamma)\|^2 + e_\gamma < T^2} f(\gamma x_0 h^{-1} x_0^{-1}) d\varrho(h)$$

$$\leq \limsup_{T \to \infty} \frac{\varrho(H_{r^3 m_0^{-1}(T^2 - M_1)^{1/2}})}{\varrho(H_T)} \cdot \frac{1}{\bar{\mu}(\Gamma\backslash G) \Delta_H(c_0)} \int_{\Gamma\backslash G} \tilde{f} d\bar{\mu}$$

$$= \frac{r^{3M}}{m_0^M \cdot \bar{\mu}(\Gamma\backslash G) \cdot \Delta_H(c_0)} \int_{\Gamma\backslash G} \tilde{f} d\bar{\mu}$$

$$= \frac{r^{3M}}{m_0^M \cdot \bar{\mu}(\Gamma\backslash G) \cdot \Delta_H(c_0)} \int_G f d\mu.$$

Now let f be arbitrary. There exists a finite cover $\mathrm{supp}(f) \subseteq \cup_i \mathcal{O}_{g_i}$. Let $c_i = c_{g_i}$ and $m_i = m_{g_i}$. Let $\alpha_i \in C_c(G)$ be a partition of unity for $\{\mathcal{O}_{g_i}\}$ such that $\sum_i \alpha_i = 1$ on $\mathrm{supp}(f)$. Then

$$\limsup_{T \to \infty} \frac{1}{\varrho(H_T)} \sum_{\gamma \in \Gamma} \int_{m_\gamma^2 \|\hat{c}_\gamma(hd_\gamma)\|^2 + e_\gamma < T^2} f(\gamma x_0 h^{-1} x_0^{-1}) d\varrho(h)$$

$$= \limsup_{T \to \infty} \frac{1}{\varrho(H_T)} \sum_{\gamma \in \Gamma} \int_{m_\gamma^2 \|\hat{c}_\gamma(hd_\gamma)\|^2 + e_\gamma < T^2} \left\{ \sum_i f(\gamma x_0 h^{-1} x_0^{-1}) \right.$$

$$\left. \cdot \alpha_i(\gamma x_0 h^{-1} x_0^{-1}) \right\} d\varrho(h) \leq \frac{r^{3M}}{\bar{\mu}(\Gamma\backslash G)} \sum_i \int_G \frac{f(g)}{m_i^M \Delta_H(c_i)} \alpha_i(g) d\mu(g)$$

$$\leq \frac{r^{3M}}{\bar{\mu}(\Gamma\backslash G)} \sum_i \int_G \left(\frac{f(g)}{m_g^M \cdot \Delta_H(c_g)} + \varepsilon f(g) \right) \alpha_i(g) d\mu(g)$$

$$= \frac{r^{3M}}{\bar{\mu}(\Gamma\backslash G)} \int_G \frac{f(g)}{m_g^M \cdot \Delta_H(c_g)} d\mu(g) + \frac{r^{3M}\varepsilon}{\bar{\mu}(\Gamma\backslash G)} \int_G f d\mu$$

for every $r > 1$ and $\varepsilon > 0$. Therefore,

$$\limsup_{T \to \infty} \frac{1}{\varrho(H_T)} \sum_{\gamma \in \Gamma} \int_{m_\gamma^2 \|\hat{c}_\gamma(hd_\gamma)\|^2 + e_\gamma < T^2} f(\gamma x_0 h x_0^{-1}) d\varrho(h)$$

$$\leq \frac{1}{\bar{\mu}(\Gamma \backslash G)} \int_G \frac{f(g)}{m_g^M \cdot \Delta_H(c_g)} d\mu(g). \tag{13}$$

Similarly, one can prove the lower bound for (8). $\qquad\square$

Proposition 9. *Let f be the characteristic function of a relatively compact Borel subset $Z \subseteq G$ such that $\mu(\partial Z) = 0$. Let $x_0 \in G$ be such that $\Gamma x_0 H$ is dense in $\Gamma \backslash G$. Then (8) holds for f.*

Proof. Denote by $\text{int}(Z)$ and $\overline{Z}$ the interior and the closure of Z respectively.

Let W_0 be an open relatively compact subset such that $\overline{Z} \subseteq W_0$. There exists $C > 0$ such that $\left(m_g^M \cdot \Delta_H(c_g)\right)^{-1} \leq C$ for $g \in W_0$.

Let $\varepsilon > 0$. There exist a compact subset $V \subseteq \text{int}(Z)$ and an open subset W such that $\overline{Z} \subseteq W \subseteq W_0$ and $\mu(W - V) < \varepsilon$. Take functions $f_1, f_2 \in C_c(G)$ such that $0 \leq f_i \leq 1$, $f_1 = 1$ on V, $f_1 = 0$ outside $\text{int}(Z)$, $f_2 = 1$ on $\overline{Z}$, and $f_2 = 0$ outside W. Then $f_1 \leq f \leq f_2$. By Proposition 8 applied to f_2,

$$\limsup_{T \to \infty} \frac{1}{\varrho(H_T)} \sum_{\gamma \in \Gamma} \int_{m_\gamma^2 \|\hat{c}_\gamma(hd_\gamma)\|^2 + e_\gamma < T^2} f(\gamma x_0 h^{-1} x_0^{-1}) d\varrho(h)$$

$$\leq \frac{1}{\bar{\mu}(\Gamma \backslash G)} \int_G \frac{f_2(g)}{m_g^M \cdot \Delta_H(c_g)} d\mu(g)$$

$$\leq \frac{1}{\bar{\mu}(\Gamma \backslash G)} \left(\int_G \frac{f(g)}{m_g^M \cdot \Delta_H(c_g)} d\mu(g) + \int_G \frac{f_2(g) - f_1(g)}{m_g^M \cdot \Delta_H(c_g)} d\mu(g) \right)$$

$$\leq \frac{1}{\bar{\mu}(\Gamma \backslash G)} \int_G \frac{f(g)}{m_g^M \cdot \Delta_H(c_g)} d\mu(g) + \frac{C\mu(W - V)}{\bar{\mu}(\Gamma \backslash G)}$$

$$\leq \frac{1}{\bar{\mu}(\Gamma \backslash G)} \int_G \frac{f(g)}{m_g^M \cdot \Delta_H(c_g)} d\mu(g) + \frac{C\varepsilon}{\bar{\mu}(\Gamma \backslash G)}$$

for every $\varepsilon > 0$. This shows (13). The dual inequality for $\liminf$ can be proved similarly. $\qquad\square$

Suppose that for a closed subset Y of G, the product map $Y \times \hat{x}_0(H) \to G$ be a homeomorphism. For $g \in G$, define $y_g \in Y$ and $h_g \in H$ such that $g = y_g \hat{x}_0(h_g)$. The map

$$\alpha : y \mapsto y \cdot \hat{x}_0(H) : Y \to G/\hat{x}_0(H).$$

is a homeomorphism too. Let ν_1 be a measure on Y such that

$$\int_G f d\mu = \int_Y \int_H f(y \hat{x}_0(h)) d\varrho(h) d\nu_1(y), \quad f \in C_c(G). \tag{14}$$

Note that such a measure exists because μ and ϱ are right invariant. Let ν be the measure on $G/\hat{x}_0(H)$ which is the image of ν_1 under α, i.e.

$$\int_Y f(\alpha(y)) d\nu_1(y) = \int_{G/\hat{x}_0(H)} f d\nu, \quad f \in C_c(G/\hat{x}_0(H)). \tag{15}$$

Note that the measure ν depends on the choice of the section Y.

Proposition 10. *Use notations as in Proposition 8. Let Ω be relatively compact Borel subset of $G/\hat{x}_0(H)$ such that $\nu(\partial\Omega) = 0$. Let*

$$N_T(\Omega) = |\{\gamma \in \Gamma : m_\gamma^2 \|\hat{c}_\gamma(h_\gamma d_\gamma)\|^2 + e_\gamma < T^2, \gamma \cdot \hat{x}_0(H) \in \Omega)\}|.$$

Then

$$\frac{N_T(\Omega)}{\varrho(H_T)} \sim \frac{1}{\bar{\mu}(\Gamma\backslash G)} \int_{\alpha^{-1}(\Omega)} \frac{1}{m_y^M \cdot \Delta_H(c_y)} d\nu_1(y)$$

as $T \to \infty$.

Proof. Let

$$\mathcal{O}_\varepsilon = \{h \in H : \|h - 1\| < \varepsilon, \|h^{-1} - 1\| < \varepsilon\}$$

for $\varepsilon > 0$.

Let ϕ be the characteristic function of $\alpha^{-1}(\Omega) \subseteq Y$. Take ψ_ε to be the characteristic function of $\mathcal{O}_\varepsilon$ normalized so that $\int_H \psi_\varepsilon d\varrho = 1$. Let

$$f_\varepsilon(g) = \phi(y_g)\psi_\varepsilon(h_g) \quad \text{for } g \in G. \tag{16}$$

Note that f satisfies conditions of Proposition 9, but before applying this proposition, we need a lemma.

Lemma 11. *For every $r > 1$, there exists $\varepsilon > 0$ such that*

$$N_{r^{-1}T}(\Omega) \leq \sum_{\gamma \in \Gamma} \int_{m_\gamma^2 \|\hat{c}_\gamma(hd_\gamma)\|^2 + e_\gamma < T^2} f_\varepsilon(\gamma x_0 h^{-1} x_0^{-1}) d\varrho(h) \leq N_{rT}(\Omega). \tag{17}$$

Proof. Note that $f_\varepsilon(\gamma x_0 h^{-1} x_0^{-1}) = 0$ for all $h \in H$ unless

$$y_\gamma \in \alpha^{-1}(\Omega), \tag{18}$$

and if the above condition holds,

$$\int_{m_\gamma^2 \|\hat{c}_\gamma(hd_\gamma)\|^2 + e_\gamma < T^2} f_\varepsilon(\gamma x_0 h^{-1} x_0^{-1}) d\varrho(h)$$

$$= \int_{m_\gamma^2 \|\hat{c}_\gamma(hd_\gamma)\|^2 + e_\gamma < T^2} \psi_\varepsilon(h_\gamma h^{-1}) d\varrho(h)$$

$$= \int_{m_\gamma^2 \|\hat{c}_\gamma(hh_\gamma d_\gamma)\|^2 + e_\gamma < T^2} \psi_\varepsilon(h^{-1}) d\varrho(h)$$

$$= \int_{m_\gamma^2 \|\hat{c}_\gamma(h)\hat{c}_\gamma(h_\gamma d_\gamma)\|^2 + e_\gamma < T^2} \psi_\varepsilon(h) d\varrho(h). \tag{19}$$

Let

$$I_\gamma \stackrel{def}{=} \int_{m_\gamma^2 \|\hat{c}_\gamma(h)\hat{c}_\gamma(h_\gamma d_\gamma)\|^2 + e_\gamma < T^2} \psi_\varepsilon(h) d\varrho(h).$$

For γ as in (18), there exists $C > 0$ such that

$$\|\hat{c}_\gamma(v)\| \leq C\|v\| \text{ and } \|\hat{c}_\gamma^{-1}(v)\| \leq C\|v\| \text{ for all } v \in \mathrm{M}(n, \mathbb{R}).$$

It follows that for every $\varepsilon > 0$,

$$\mathcal{O}_{\varepsilon/C} \subseteq \hat{c}_\gamma(\mathcal{O}_\varepsilon) \subseteq \mathcal{O}_{C\varepsilon}.$$

Therefore, by Lemma 7, there exists $\varepsilon > 0$ such that

$$r^{-1}\|v\| \leq \|\hat{c}_\gamma(h)v\| \leq r\|v\| \tag{20}$$

for every $v \in \mathrm{M}(n,\mathbb{R})$, $h \in \mathcal{O}_\varepsilon$, and $\gamma \in \Gamma$ such that (18) holds.

Let $\gamma \in \Gamma$ be such that

$$m_\gamma^2\|\hat{c}_\gamma(h_\gamma d_\gamma)\|^2 + e_\gamma < r^{-2}T^2,$$

and (18) holds. Then by (20),

$$m_\gamma^2\|\hat{c}_\gamma(h)\hat{c}_\gamma(h_\gamma d_\gamma)\|^2 + e_\gamma \leq r^2 m_\gamma^2\|\hat{c}_\gamma(h_\gamma d_\gamma)\|^2 + e_\gamma < T^2$$

for $h \in \mathcal{O}_\varepsilon$. It follows that the $I_\gamma = 1$. This proves the first inequality in (17).

Note that $I_\gamma \leq 1$. Let $\gamma \in \Gamma$ be such that $I_\gamma \neq 0$. Then (18) holds, and for some $h \in \mathcal{O}_\varepsilon$,

$$m_\gamma^2\|\hat{c}_\gamma(h)\hat{c}_\gamma(h_\gamma d_\gamma)\|^2 + e_\gamma < T^2.$$

Using (20), we deduce that

$$m_\gamma^2\|\hat{c}_\gamma(h_\gamma d_\gamma)\|^2 + e_\gamma < r^2T^2.$$

This proves the second inequality in (17). $\qquad\square$

Now we can use Proposition 9 to find asymptotics for $N_T(\Omega)$. By Lemma 11, for every $r > 1$ there exists $\varepsilon > 0$ such that

$$\limsup_{T\to\infty} \frac{N_T(\Omega)}{\varrho(H_T)}$$

$$\leq \limsup_{T\to\infty} \frac{1}{\varrho(H_T)} \sum_{\gamma\in\Gamma} \int_{m_\gamma^2\|\hat{c}_\gamma(hd_\gamma)\|^2+e_\gamma<r^2T^2} f_\varepsilon(\gamma x_0 h^{-1}x_0^{-1})d\varrho(h).$$

Therefore, by Proposition 9, (5), (14), and (15),

$$\limsup_{T\to\infty} \frac{N_T(\Omega)}{\varrho(H_T)}$$

$$\leq \left(\limsup_{T\to\infty} \frac{\varrho(H_{rT})}{\varrho(H_T)}\right) \frac{1}{\bar{\mu}(\Gamma\backslash G)} \int_G \frac{f_\varepsilon(g)}{m_g^M \cdot \Delta_H(c_g)}d\mu(g)$$

$$= \frac{r^M}{\bar{\mu}(\Gamma\backslash G)} \int_{Y\times H} \frac{f_\varepsilon(y\hat{x}_0(h))}{m_y^M \cdot \Delta_H(c_y)}d\nu_1(y)d\varrho(h)$$

$$= \frac{r^M}{\bar{\mu}(\Gamma\backslash G)} \int_Y \frac{\phi(y)}{m_y^M \cdot \Delta_H(c_y)}d\nu_1(y) \int_H \psi_\varepsilon(h)d\varrho(h)$$

$$= \frac{r^M}{\bar{\mu}(\Gamma\backslash G)} \int_{\alpha^{-1}(\Omega)} \frac{d\nu_1(y)}{m_y^M \cdot \Delta_H(c_y)}.$$

Taking $r \to 1+$, we get

$$\limsup_{T\to\infty} \frac{N_T(\Omega)}{\varrho(H_T)} \leq \frac{1}{\bar{\mu}(\Gamma\backslash G)} \int_{\alpha^{-1}(\Omega)} \frac{d\nu_1(y)}{m_y^M \cdot \Delta_H(c_y)}.$$

Similarly, one can prove that

$$\liminf_{T\to\infty} \frac{N_T(\Omega)}{\varrho(H_T)} \geq \frac{1}{\bar{\mu}(\Gamma\backslash G)} \int_{\alpha^{-1}(\Omega)} \frac{d\nu_1(y)}{m_y^M \cdot \Delta_H(c_y)}.$$

This proves the Proposition. $\square$

3. Uniform distribution for a lattice in $\mathrm{SL}(n, \mathbb{R})$

3.1. Density of orbits.

In this section we derive Theorems 1 and 2 from the following result on topological rigidity of unipotent flow, which was proved by M. Ratner:

Theorem 12. (Ratner [Rat91b]**)** *Let G be a connected Lie group, Γ be a lattice in G, and U be a subgroup of G generated by Ad-unipotent 1-parameter subgroups. Then for every $x \in \Gamma \backslash G$, $\overline{xU} = xH$, where H is a closed connected subgroup of G such that $U \subseteq H$, and xH supports finite H-invariant Borel measure.*

Note that the proofs of Dani, Raghavan, Veech are different from the proofs that are presented here. In fact, their proofs can be considered as the first important steps towards the general result on topological rigidity – Theorem 12.

We start the proof of Theorem 1 with a simple lemma:

Lemma 13. *Let $\{v_i : i = 1, \ldots, s\} \subseteq \mathbb{R}^n$, $1 \le s \le n - 2$, be linearly independent vectors such that $\langle v_i : i = 1, \ldots, s \rangle$ contains no nonzero rational vectors. Then there exists v_{s+1} such that v_i, $i = 1, \ldots, s + 1$, are linearly independent, and $\langle v_i : i = 1, \ldots, s + 1 \rangle$ contains no nonzero rational vectors.*

Proof. Let $V = \langle v_i : i = 1, \ldots, s \rangle$. Since $s \le n - 2$, for any $v \in \mathbb{R}^n$, $\langle V, v \rangle$ is a proper subspace of $\mathbb{R}^n$. Therefore one can take a vector v_{s+1} outside $\cup_{v \in \mathbb{Q}^n} \langle V, v \rangle$. If $v = \sum_{i=1}^{s+1} \alpha_i v_i$ is rational and nonzero for some $\alpha_i \in \mathbb{R}$, then $\alpha_{s+1} \ne 0$, and $v_{s+1} \in \langle V, v \rangle$. This is a contradiction. Thus, v_{s+1} is as required. $\square$

Proof of Theorem 1. It is easy to see that if the condition of the theorem is not satisfied, the orbit cannot be dense. The hard part is to show that the above condition implies density. By Lemma 13, we may assume that $l = n - 1$.

Denote $G = \mathrm{SL}(n, \mathbb{R})$, $\Gamma = \mathrm{SL}(n, \mathbb{Z})$, and

$$U_0 = \left(\begin{array}{c|c} E & * \\ \hline 0 & 1 \end{array} \right),$$

where E is the identity $(n - 1) \times (n - 1)$ matrix. Let $g_0 \in G$ be such that $g_0 v_i = e_i$ for $i = 1, \ldots, n - 1$. Here $\{e_i : i = 1, \ldots, n\}$ is the standard basis of $\mathbb{R}^n$. Then the stabilizer of v in G is $U = g_0^{-1} U_0 g_0$. Note that any nontrivial U-invariant subspace of $\mathbb{R}^n$ is contained in $\langle v_i : i = 1, \ldots, n - 1 \rangle$. Consider U-orbit $\Gamma U \subset \Gamma \backslash G$. By Ratner's theorem (Theorem 12), $\overline{\Gamma U} = \Gamma H$ where H is a closed connected subgroup of G containing U, and $H \cap \Gamma$ is a lattice in H. Moreover by [Sh91, Proposition 3.2], H is the connected component of the smallest real algebraic $\mathbb{Q}$-subgroup containing U, and the radical of H is unipotent. Let R be the radical of H. Since R is defined over $\mathbb{Q}$ and unipotent, the space V^R of R-fixed vectors is nonzero and defined over $\mathbb{Q}$. Also V^R is H-invariant because R is normal in H. Thus, if $V^R \ne \mathbb{R}^n$, $V^R \subseteq \langle v_i : i = 1, \ldots, n - 1 \rangle$. However, this contradicts our hypothesis on v. Therefore, $V^R = \mathbb{R}^n$ and $R = 1$, i.e. H is semisimple. We claim that $H = G$. To simplify notations, we work with the group $H_0 \stackrel{def}{=} g_0 H g_0^{-1}$. Let $\mathfrak{h}$ and $\mathfrak{u}$ be the Lie algebras of H_0

and U_0 respectively. Since $U_0 \subseteq H_0$,

$$\mathfrak{u} = \langle E_{in} : i = 1, \ldots, n-1 \rangle \subseteq \mathfrak{h}.$$

Here E_{ij} denotes a matrix with 1 at the place (i, j) and 0 elsewhere. Using that the Killing form $k(x, y) = \operatorname{Tr}(xy)$ for $x, y \in \mathfrak{sl}(n, \mathbb{R})$ is nondegenerate on $\mathfrak{h}$, one shows that the projection map from $\mathfrak{h}$ to the space $\langle E_{ni} : i = 1, \ldots, n-1 \rangle$ with respect to the basis $\{E_{ij}\}$ is surjective. Thus for $i = 1, \ldots, n-1$, there exists $h_i = E_{ni} + \tilde{h}_i \in \mathfrak{h}$ with $\tilde{h}_i$ in the normalizer of $\mathfrak{u}$. Then

$$\mathfrak{h} \supseteq [h_i, \mathfrak{u}] + \mathfrak{u} \supseteq [E_{ni}, \mathfrak{u}], \quad i = 1, \ldots, n-1.$$

It follows that $\mathfrak{h} = \mathfrak{sl}(n, \mathbb{R})$ and $H = G$. Thus, $\overline{\Gamma U} = G$. Finally,

$$\overline{\Gamma v} = \overline{\Gamma U v} \supseteq \overline{\Gamma U} v = G v = \mathcal{V}_{n,l}. \tag{21}$$

$\square$

Proof of Theorem 2. It is sufficient to prove the claim for $l = n - 1$.

Let U be as in the proof of Theorem 1. By (21), we just need to show that ΓU is dense in G. By Ratner's theorem (Theorem 12), $\overline{\Gamma U} = \Gamma H$ where H is a closed connected subgroup of G containing U, and $H \cap \Gamma$ is a lattice in H. By Lemma 3.8 and Proposition 3.10 from [Sh91], one of the following two possibilities holds: H is reductive, or $W \cap \Gamma$ is a lattice in W where W is the unipotent radical of a proper parabolic subgroup of G. Since Γ is cocompact, it follows from Godement's criterion that Γ has no nontrivial unipotent elements. This contradicts the second possibility. Thus, H is reductive, and the Killing form is nondegenerate on the Lie algebra of H. Now one can show by the same argument as in the proof of Theorem 1 that $H = G$. Hence, ΓU is dense in $\Gamma \backslash G$. This implies Theorem 2. $\square$

3.2. Iwasawa decomposition for $\mathrm{SL}(n, \mathbb{R})$.

Fix $l = 1, \ldots, n - 1$.

For $s = (s_1, \ldots, s_n) \in \mathbb{R}^n$, $\sum_{i=1}^n s_i = 0$, define

$$a(s) = \operatorname{diag}(e^{s_1}, \ldots, e^{s_n}) \in \mathrm{SL}(n, \mathbb{R}).$$

For a vector s as above, define decomposition

$$s = s^- + s^+$$

with $s^- = (s_1, \ldots, s_l, r, \ldots, r)$, $r = \frac{1}{n-l}(-s_1 - \cdots - s_l)$, $s^+ = s - s^-$. Note that r is chosen so that $a(s^+), a(s^-) \in \mathrm{SL}(n, \mathbb{R})$.

For $t = (t_{ij} : 1 \le i < j \le l)$, $t_{ij} \in \mathbb{R}$, denote by $n^-(t)$ the unipotent upper triangular matrix which entries above diagonal are equal t_{ij} for $i < j \le l$ and 0 otherwise. Similarly, for $t = (t_{ij} : 1 \le i < j \le n, j > l)$, $t_{ij} \in \mathbb{R}$, denote by $n^+(t)$ the unipotent upper triangular matrix which entries above diagonal are equal t_{ij} for $1 \le i < j \le n$, $j > l$ and 0 otherwise.

We use the following notations:

$$
\begin{aligned}
G &= \mathrm{SL}(n, \mathbb{R}), \\
K &= \mathrm{SO}(n, \mathbb{R}), \\
A^o_{l-} &= \left\{ a(s^-) \ : \ s \in \mathbb{R}^n, \ \sum_{i=1}^n s_i = 0 \right\} \\
A^o_{l+} &= \left\{ a(s^+) \ : \ s \in \mathbb{R}^n, \ \sum_{i=1}^n s_i = 0 \right\}, \\
A^o &= A^o_{l-} A^o_{l+} = \left\{ a(s) \ : \ s \in \mathbb{R}^n, \ \sum_{i=1}^n s_i = 0 \right\}, \\
N_{l-} &= \{ n^-(t) \ : \ t_{ij} \in \mathbb{R}, \ 1 \le i < j \le l \}, \\
N_{l+} &= \{ n^+(t) \ : \ t_{ij} \in \mathbb{R}, \ 1 \le i < j \le n, j > l \}, \\
N &= N_{l-} N_{l+} = \text{``unipotent upper triangular group''}, \\
B^o_l &= A^o_{l+} N_{l+} = N_{l+} A^o_{l+}.
\end{aligned}
$$

Denote by dk the normalized Haar measure on K.
Let

$$
dn^+ = dt^+ = \prod_{i < j \le l} dt_{ij} \quad \text{and} \quad dn^- = dt^- = \prod_{\max(i,l) < j} dt_{ij}.
$$

These measures are Haar measures for N_{l+} and N_{l-} respectively. The subgroup N_{l+} is normal in N, and the product map

$$
N_{l-} \times N_{l+} \to N
$$

is a diffeomorphism. Also the image of product of dn^- and dn^+ under this map is a Haar measure on N. Let us denote it by dn:

$$
\int_N f(n)dn = \int_{N_{l-} \times N_{l+}} f(n^- n^+) dn^- dn^+, \quad f \in C_c(N). \tag{22}
$$

Haar measures on A^o_{l-} and A^o_{l+} are defined by

$$
da^- = ds^- = \prod_{i=1}^l ds_i^- \quad \text{and} \quad da^+ = ds^+ = \prod_{i=l+1}^{n-1} ds_i^+
$$

respectively. Then a Haar measure da on $A^o = A^o_{l-} A^o_{l+}$ is the product measure:

$$
\int_{A^o} f(a)da = \int_{A^o_{l-} \times A^o_{l+}} f(a^- a^+) da^- da^+, \quad f \in C_c(A^o). \tag{23}
$$

The product map $A^o_{l+} \times N_{l+} \to B^o_l$ is a diffeomorphism, and the image of the product measure under this map is a left Haar measure on B^o_l. Denote this measure by λ_l. Then a right Haar measure ϱ_l on B^o_l can be defined by

$$
\varrho_l(f) = \int_{B^o_l} f(b^{-1})\lambda_l(b) = \int_{A^o_{l+} \times N_{l+}} f(a(s^+)n^+)\delta_l^+(s^+)ds^+ dn^+ \tag{24}
$$

for $f \in C_c(B_l^o)$, where

$$\delta_l^+(a) = \delta_l^+(s) = \exp\left\{2\sum_{i=l+1}^{n}(n-i)s_i\right\} \tag{25}$$

for $a = \mathrm{diag}(e^{s_1}, \ldots, e^{s_n}) \in \mathrm{SL}(n, \mathbb{R})$.

By Iwasawa decomposition, the map

$$(k, n, a) \mapsto kna : K \times N \times A^o \to G \tag{26}$$

is a diffeomorphism.

Lemma 14. *Let $e^0 = (e_1, \ldots, e_l)$ be the standard orthonormal frame in $\mathbb{R}^n$. Then for $k \in K$, $n \in N$, and $a = \mathrm{diag}(e^{s_1}, \ldots, e^{s_n}) \in A^o$,*

$$\mathrm{Vol}(knae^0) = \exp\left\{\sum_{i=1}^{l}s_i\right\}.$$

Proof. Let $g = kna$. Recall that Iwasawa decomposition is proved using Gramm-Schmidt orthogonalization for basis $v_i = ge_i$. Let $\{w_i\}$ be an orthonormal basis such that $\langle v_k : k \leq i\rangle = \langle w_k : k \leq i\rangle$ for $1 \leq i \leq n$. Then $e^{s_i} = v_i \cdot w_i$, i.e. e^{s_i} is the length of projection of v_i onto the orthogonal complement of $\langle v_k : k \leq i - 1\rangle$ in $\langle v_k : k \leq i\rangle$. Now the statement is obvious. $\square$

Define

$$\delta_l^-(a) = \delta_l^-(s) = \exp\left\{\sum_{i=1}^{l}s_i\right\}, \tag{27}$$

where $a = \mathrm{diag}(e^{s_1}, \ldots, e^{s_n}) \in \mathrm{SL}(n, \mathbb{R})$.

The image of the product measure under the map (26) is a Haar measure on G [He, Proposition X.1.12]. Let us denote this image by μ:

$$\int_G f d\mu = \int_{K \times N \times A^o} f(kna)\,dk\,dn\,da, \quad f \in C_c(G). \tag{28}$$

For a lattice Γ in G, there exists a measure $\bar{\mu}$ on $\Gamma \backslash G$ such that

$$\int_G f d\mu = \int_{\Gamma \backslash G} \sum_{\gamma \in \Gamma} f(\gamma g)\,d\bar{\mu}(g), \quad f \in C_c(G). \tag{29}$$

For our purposes, we modify the Iwasawa decomposition as follows:

$$(k, n^-, a^-, b) \mapsto kn^-a^-b : K \times N_{l-} \times A_{l-}^o \times B_l^o \to G. \tag{30}$$

Since A_{l-}^o normalizes B_l^o, this map is a diffeomorphism too.

Fix $g_0 \in G$. By (30) the map

$$(k, n^-, a^-, b) \mapsto kn^-a^- g_0 b : K \times N_{l-} \times A_{l-}^o \times (B_l^o)^{g_0} \to G \tag{31}$$

is a diffeomorphism. Here we use notation: $X^{g_0} = g_0^{-1}Xg_0$.

For $g \in G$ define $k_g \in K$, $a_g^\pm \in A_{l\pm}^o$, and $n_g^\pm \in N_{l\pm}$ such that

$$g = k_g n_g^- a_g^- g_0 (a_g^+ n_g^+)^{g_0}. \tag{32}$$

Also define

$$b_g^- = n_g^- a_g^-, \quad b_g = a_g^+ n_g^+,$$

and $b'_g \in \mathrm{GL}(l, \mathbb{R})$ such that

$$b_g^- = \left(\begin{array}{c|c} b'_g & 0 \\ \hline 0 & * \end{array} \right). \tag{33}$$

Lemma 15. *For $f \in C_c(G)$ and $g_0 \in G$,*

$$\int_G f d\mu = \int_{K \times N_{l-} \times A^o_{l-} \times B^o_l} f(kn^- a(s^-) g_0 b^{g_0}) \delta_l^-(s^-)^n dk dn^- ds^- d\varrho_l(b),$$

where δ_l^- is defined in (27).

Proof. By (28), (22), and (23),

$$\int_G f d\mu = \int_{K \times N_{l-} \times N_{l+} \times A^o_{l-} \times A^o_{l+}} f(kn^- n^+ a^- a^+) dk dn^- dn^+ da^- da^+. \tag{34}$$

The Jacobian of the map

$$N_{l+} \to N_{l+} : n \mapsto a^{-1} n a$$

for $a = a(s^-) a(s^+)$ is equal to $\delta_l^-(s^-)^{-n} \delta_l^+(s^+)^{-1}$. Thus, it follows from (24) and (34) that

$$\int_G f d\mu = \int_{K \times N_{l-} \times A^o_{l-} \times B^o_l} f(kn^- a(s^-) b) \delta_l^-(s^-)^n dk dn^- ds^- d\varrho_l(b).$$

Then since G is unimodular,

$$\begin{aligned} \int_G f d\mu &= \int_G f(g g_0) d\mu(g) \\ &= \int_{K \times N_{l-} \times A^o_{l-} \times B^o_l} f(kn^- a(s^-) b g_0) \delta_l^-(s^-)^n dk dn^- ds^- d\varrho_l(b) \\ &= \int_{K \times N_{l-} \times A^o_{l-} \times B^o_l} f(kn^- a(s^-) g_0 b^{g_0}) \delta_l^-(s^-)^n dk dn^- ds^- d\varrho_l(b). \end{aligned}$$

$\square$

Lemma 16. *Let $e^0 = (e_1, \ldots, e_l)$ be the standard frame in $\mathbb{R}^n$. For $f \in C_c(\mathcal{V}_{n,l})$,*

$$\int_{\mathcal{V}_{n,l}} f(v) dv = d_{n,l} \int_{K \times N_{l-} \times A^o_{l-}} f(kn^- a(s^-) e^0) \delta_l^-(s^-)^n dk dn^- ds^-, \tag{35}$$

where $d_{n,l}$ is a constant computed in (73).

Proof. The measure on the left side of (35) is G-invariant. We claim that the measure on the right side is G-invariant too. It is easy to see that the map

$$g B^o_l \mapsto g e^0 : G/B^o_l \to \mathcal{V}_{n,l}$$

is proper. Thus, every function $f \in C_c(\mathcal{V}_{n,l})$ can be lifted to a function $f_1 \in C_c(G/B^o_l)$. Then the function f can be represented as

$$f(g e^0) = f_1(g B^o_l) = \int_{B^o_l} f_2(g b) d\varrho_l(b)$$

for some $f_2 \in C_c(G)$ (see [Rag, Ch. 1]). Then by Lemma 15,

$$\int_{K \times N_{l-} \times A_{l-}^o} f(kn^- a(s^-)e^0)\delta_l^-(s^-)^n dk dn^- ds^- = \int_G f_2 d\mu.$$

It follows that the measure on the right side of (35) is G-invariant. By uniqueness of Haar measure, the integrals are equal up to a scalar multiple $d_{n,l}$. This constant is computed in the Appendix. $\qquad \square$

3.3. **Volume estimates.**

For a set $S \subseteq G$ and $T > 0$, define

$$S_T = \{s \in S : \|s\| < T\}.$$

We compute the asymptotics of $\varrho_l(B_{l,T}^o)$ as $T \to \infty$:

Lemma 17.
$$\varrho_l(B_{l,T}^o) \sim \gamma_{n,l} T^{(n-1)(n-l)} \quad as \quad T \to \infty, \tag{36}$$
where the constant $\gamma_{n,l}$ is given in (80).

The proof, which is given in the Appendix, follows the method of Duke, Rudnick, Sarnak [DRS93].

For $C \in \mathbb{R}$, define

$$\begin{aligned} A_{l+}^C &= \{a(s^+) : s_i^+ > C, i = l+1, \ldots, n-1\}, \\ B_l^C &= A_{l+}^C N_{l+}. \end{aligned}$$

The following "measure concentration" result plays a crucial role in our proof.

Lemma 18. *For* $C \in \mathbb{R}$,
$$\varrho_l(B_{l,T}^C) \sim \varrho_l(B_{l,T}^o) \quad as \ T \to \infty.$$

This lemma is proved in the Appendix.

3.4. **Uniform distribution.**

Theorem 19. *Let* Γ *be a lattice in* $G = SL(n, \mathbb{R})$. *Fix* $g_0 \in (KN_{l-}A_{l-}^o)^{-1}$ *such that* $\Gamma(B_l^o)^{g_0}$ *is dense in* G. *Let* $Y = KN_{l-}A_{l-}^o g_0$, *and* ν_1 *be a measure on* Y *such that*

$$\int_G f d\mu = \int_Y \int_{B_l^o} f(yb^{g_0}) d\varrho_l(b) d\nu_1(y), \quad f \in C_c(G). \tag{37}$$

Let ν *be a measure on* $G/(B_l^o)^{g_0}$ *induced by* ν_1. *For* $T > 0$ *an* $\Omega \subseteq G/(B_l^o)^{g_0}$, *denote*

$$N_T(\Omega, g_0) = |\{\gamma \in \Gamma : \|\gamma\| < T, \gamma(B_l^o)^{g_0} \in \Omega\}|. \tag{38}$$

Then for relatively compact Borel subset Ω *of* $G/(B_l^o)^{g_0}$ *such that* $\nu(\partial\Omega) = 0$,

$$N_T(\Omega, g_0) \sim \varrho(B_{l,T}^o) \frac{\delta_l^-(a_0)^{1-n}}{\bar{\mu}(\Gamma \backslash G)} \int_\Omega \frac{d\nu(x)}{\delta_l^-(a_x^-)} \quad as \quad T \to \infty,$$

where a_0 *and* a_x^- *are the* A_{l-}^o-*components of* g_0^{-1} *and* x *with respect to decomposition (30) respectively.*

Note that a similar result holds for every $g_0 \in G$ because of the decomposition (30).

Proof. Write $g_0^{-1} = k_0 b_0^-$ for $k_0 \in K$, $b_0^- = n_0 a_0 \in N_{l-} A_{l-}^o$.

It is convenient to use decomposition (32). The product map $Y \times (B_l^o)^{g_0} \to G$ is a diffeomorphism. For $g \in G$, denote $y_g = k_g n_g^- a_g^- g_0$, the Y-component of g. The map

$$\alpha : Y \to G/(B_l^o)^{g_0} : y \mapsto y(B_l^o)^{g_0}$$

is a diffeomorphism. Clearly, $\gamma(B_l^o)^{g_0} \in \Omega$ iff $y_\gamma \in \alpha^{-1}(\Omega)$.

For $g \in G$,

$$\|g\| = \|k_g b_g^- b_g g_0\| = \|b_g^- b_g (b_0^-)^{-1}\|. \tag{39}$$

Note that for $\left(\begin{array}{c|c} * & X \\ \hline 0 & Y \end{array} \right) \in \mathrm{SL}(n, \mathbb{R})$,

$$b_g^- \left(\begin{array}{c|c} * & X \\ \hline 0 & Y \end{array} \right) (b_0^-)^{-1} = \left(\begin{array}{c|c} * & \beta_0^{-1}(b_g' \cdot X) \\ \hline 0 & \beta_0^{-1} \beta_g Y \end{array} \right), \tag{40}$$

where

$$\beta_0 = \det(b_0')^{-\frac{1}{n-l}} \quad \text{and} \quad \beta_g = \det(b_g')^{-\frac{1}{n-l}} \tag{41}$$

(here b_0' and b_g' are defined as in (33)). Put

$$c_g = b_g^- \quad \text{and} \quad m_g = |\beta_0^{-1} \beta_g|. \tag{42}$$

It follows from (39) and (40) that for $g \in G$,

$$\|g\|^2 = m_g^2 \|\hat{c}_g(b_g)\|^2 + e_g,$$

where e_g is a continuous function depending only on the b^--components of g with respect to decomposition (32).

Using previous notations, we have

$$N_T(\Omega, g_0) = |\{\gamma \in \Gamma : m_\gamma^2 \|\hat{c}_\gamma(b_\gamma)\|^2 + e_\gamma < T^2, y_\gamma \in \alpha^{-1}(\Omega)\}|. \tag{43}$$

To derive asymptotics of $N_T(\Omega, g_0)$, we can use Proposition 10 with $H = B_l^o$, $h_\gamma = b_\gamma$, and $d_\gamma = e$. Note that by Lemma 17 the condition (5) for $H = B_l^o$ is satisfied with $M = (n-1)(n-l)$, and by Theorem 20 below, the condition (6) holds. Therefore, applying Proposition 10, we get

$$N_T(\Omega, g_0) \sim \frac{\varrho(B_{l,T}^o)}{\bar{\mu}(\Gamma \backslash G)} \int_{\alpha^{-1}(\Omega)} \frac{1}{m_y^{(n-1)(n-l)} \cdot \Delta_H(c_y)} d\nu_1(y)$$

as $T \to \infty$, where Δ_H is defined in (7). By (41) and (42),

$$m_g = (\delta_l^-(a_0)^{-1} \delta_l^-(a_g^-))^{-\frac{1}{n-l}}.$$

Also

$$\Delta_{B_l^o}(c_g) = \frac{\det(b_g')^{n-l}}{\beta_g^{l(n-l)}} = \delta_l^-(a_g^-)^n.$$

Thus,

$$\int_{\alpha^{-1}(\Omega)} \frac{1}{m_y^{(n-1)(n-l)} \cdot \Delta_H(c_y)} d\nu_1(y) = \int_{\alpha^{-1}(\Omega)} \frac{\delta_l^-(a_0)^{1-n}}{\delta_l^-(a_y^-)} d\nu_1(y)$$

$$= \delta_l^-(a_0)^{1-n} \int_\Omega \frac{d\nu(x)}{\delta_l^-(a_x^-)}.$$

This proves the theorem.   □

Proof of Theorem 3. For some $g_0 \in (KN_{l-}A^o_{l-})^{-1}$, $v^0 = g_0^{-1}e^0$ where $e^0 = (e_1, \ldots, e_l)$ is the standard frame. The condition that Γv^o is dense in $\mathcal{V}_{n,l}$ is equivalent to $\Gamma G_l^{g_0}$ is dense in G where

$$G_l = \left(\begin{array}{c|c} E & * \\ \hline 0 & \mathrm{SL}(n-l, \mathbb{R}) \end{array} \right).$$

Since B^o_l is epimorphic in G_l, it follows from [SW00, Corollary 1.3] that $\Gamma(B^o_l)^{g_0}$ is dense in G.

Consider a map

$$\alpha : G/(B^o_l)^{g_0} \to \mathcal{V}_{n,l} \simeq G/(G_l)^{g_0} : g(B^o_l)^{g_0} \mapsto gv^0.$$

Note that this map is proper and G-equivariant. Put $\Omega^* = \alpha^{-1}(\Omega)$. Then Ω^* is relatively compact, and $N_T(\Omega, v^0) = N_T(\Omega^*, g_0)$, where $N_T(\Omega^*, g_0)$ is defined in (38).

Let ν be the measure on $G/(B^o_l)^{g_0}$ defined in (37). It follows from Lemmas 15 and 16 that $\alpha(\nu) = d^{-1}_{n,l}dv$, where $d_{n,l}$ is defined in (73) and dv is the Lebesgue measure on $\mathcal{V}_{n,l}$.

One can check that $\alpha(\partial\Omega^*) \subseteq \partial\Omega$. Therefore,

$$\nu(\partial\Omega^*) \leq \nu(\alpha^{-1}(\partial\Omega)) = d^{-1}_{n,l} \int_{\partial\Omega} dv = 0.$$

By Theorem 19,

$$N_T(\Omega, v^0) \sim \varrho(B^o_{l,T}) \frac{\delta^-_l(a_0)^{1-n}}{\bar{\mu}(\Gamma\backslash G)} \int_{\Omega^*} \frac{d\nu(x)}{\delta^-_l(a^-_x)} \quad \text{as} \quad T \to \infty.$$

By Lemma 14, $\delta^-_l(a_0) = \mathrm{Vol}(v^0)$. Using decomposition (32), we have $gv^0 = k_g n^-_g a^-_g e^0$. Thus, by Lemma 14, $\delta^-_l(a^-_g) = \mathrm{Vol}(gv^0)$. Hence,

$$N_T(\Omega, v^0) \sim \varrho(B^o_{l,T}) \frac{\mathrm{Vol}(v^0)^{1-n}}{d_{n,l}\bar{\mu}(\Gamma\backslash G)} \left(\int_\Omega \frac{dv}{\mathrm{Vol}(v)} \right) \quad \text{as} \quad T \to \infty.$$

Finally, using Lemma 17 and (73), we have

$$N_T(\Omega, v^0) \sim a_{n,l} \frac{\mathrm{Vol}(v^0)^{1-n}}{\bar{\mu}(\Gamma\backslash G)} \left(\int_\Omega \frac{dv}{\mathrm{Vol}(v)} \right) T^{(n-1)(n-l)} \quad \text{as} \quad T \to \infty, \tag{44}$$

where

$$a_{n,l} = \frac{\gamma_{n,l}}{d_{n,l}}. \tag{45}$$

The constants $\gamma_{n,l}$ and $d_{n,l}$ are computed in the Appendix.   □

The proof of Corollary 4 is presented in the Appendix.

4. Ergodic Theorem

The main result of this section is the following ergodic theorem along balls in B^o_l.

Theorem 20. *Let $1 \le l \le n-1$. Let Γ be a lattice in G, and $y \in \Gamma \backslash G$ be such that yB_l^o is dense in $\Gamma \backslash G$. Denote by ν the probability G-invariant measure on $\Gamma \backslash G$. Then for any $\tilde{f} \in C_c(\Gamma \backslash G)$,*

$$\frac{1}{\varrho_l(B_{l,T}^o)} \int_{B_{l,T}^o} \tilde{f}(yb^{-1})d\varrho_l(b) \to \int_{\Gamma \backslash G} \tilde{f}d\nu \quad as \quad T \to \infty.$$

If $l = n-1$, the group B_l^o is unipotent, and Theorem 20 is a special case of the result of Shah [Sh94]. Thus, we may assume that $l < n-1$.

4.1. **Representations of** $\mathrm{SL}(n, \mathbb{R})$.

Before starting the proof, we prepare some auxiliary results on representations of $G = \mathrm{SL}(n, \mathbb{R})$.

Denote by $\mathfrak{g}$ the Lie algebra of G. We have the root space decomposition of $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes \mathbb{C}$:

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}_0 \oplus \sum_{i \neq j} \mathbb{C}E_{ij},$$

where $\mathfrak{g}_0$ is the diagonal subalgebra of $\mathfrak{g}_{\mathbb{C}}$, and E_{ij} is the matrix with 1 in position (i, j) and 0's elsewhere. It is convenient to identify $\mathfrak{g}_0$ with the space of vectors $s = (s_1, \ldots, s_n) \in \mathbb{C}^n$, $\sum_i s_i = 0$. Introduce the *roots* of $\mathfrak{g}_{\mathbb{C}}$:

$$\alpha_{ij}(s) = s_i - s_j, \quad i \neq j,$$

and the *fundamental weights* of $\mathfrak{g}_{\mathbb{C}}$:

$$\omega_i(s) = s_1 + \cdots + s_i, \quad 1 \le i \le n-1.$$

The *simple roots* of $\mathfrak{g}_{\mathbb{C}}$ are $\alpha_i = \alpha_{i,i+1}$, $i = 1, \ldots, n-1$. For $i < j$,

$$\alpha_{ij} = \alpha_i + \cdots + \alpha_j.$$

The *dominant weights* are linear combinations with nonnegative integer coefficients of the fundamental weights. A *highest weight* of a finite-dimensional representation of $\mathfrak{g}_{\mathbb{C}}$ is a weight that is maximal with respect to the ordering on the dual space of $\mathfrak{g}_0$. This weight is unique. Irreducible representations of $\mathfrak{g}_{\mathbb{C}}$ are in one-to-one correspondence with the dominant weights. The corresponding dominant weight is the highest weight of the representation.

Let $\mathfrak{g}_0^+$ be the Lie subalgebra of $\mathfrak{g}_{\mathbb{C}}$ that corresponds to A_{l+}^o. That is, $\mathfrak{g}_0^+$ consists of diagonal matrices with entries

$$(0, \ldots, 0, s_{l+1}, \ldots, s_n), \quad s_{l+1} + \cdots + s_n = 0,$$

on the diagonal.

Lemma 21. *Let π be a representation of G on a finite-dimensional complex vector space V. Let*

$$V_0 = \{v \in V : \pi(B_l^o)v = v\}. \tag{46}$$

Then every vector $\bar{v} \in V/V_0 - \{0\}$ such that $\pi(N_{l+})\bar{v} = \bar{v}$ is a sum of weight vectors of $\mathfrak{g}_0^+$ with nonzero dominant weights.

Proof. First, we show that there are no nonzero vectors in V/V_0 fixed by B_l^o. Let W be the maximal B_l^o-invariant subspace on which B_l^o acts unipotently. The space W can be constructed inductively as follows. Let W_0 be the space of vectors fixed by B_l^o, $W_1 \supseteq W_0$ be the space such that W_1/W_0 is the space of vectors in V/W_0 fixed by B_l^o, and so on. After finitely many steps, we get W. We claim that $W = V_0$. Note A_{l+}^o acts trivially on W. Take $w \in W$. Suppose that $\tilde{\pi}(E_{ij})w \neq 0$ for some $E_{ij} \in \mathfrak{n}^+$, where $\mathfrak{n}^+$ is the Lie algebra of N_{l+}. Then it is a weight vector with weight $\alpha_{ij}|_{\mathfrak{g}_0^+} \neq 0$ with respect to $\mathfrak{g}_0^+$. This is a contradiction. Thus, $\tilde{\pi}(\mathfrak{n}^+)w = 0$ for every $w \in W$, and $W = V_0$. Clearly, the space V/W does not contain any vectors that are fixed by B_{l+}^o. This proves the claim.

Let

$$\tilde{G} = \left(\begin{array}{c|c} E & 0 \\ \hline 0 & \mathrm{SL}(n-l, \mathbb{R}) \end{array} \right) \quad \text{and} \quad \tilde{N} = \tilde{G} \cap N_{l+}.$$

Since $A_{l+}^o \tilde{N}$ is an epimorphic subgroup of $\tilde{G}$, the space V_0 is $\tilde{G}$-invariant. Take a vector $\bar{v} \in V/V_0 - \{0\}$ such that $\pi(N_{l+})\bar{v} = \bar{v}$. By [Go02, Lemma 15] applied to $\tilde{G}$, $\bar{v} = \sum_{k=1}^m \bar{v}_k$ where $\bar{v}_k$, $k = 1, \ldots, m$, are weight vectors with dominant weights λ_k with respect to $\mathfrak{g}_0^+$. Without loss of generality, $\lambda_k \neq \lambda_j$ for $k \neq j$. For $E_{ij} \in \mathfrak{n}^+$, we have $\sum_{k=1}^m \tilde{\pi}(E_{ij})\bar{v}_k = 0$. Using that $\tilde{\pi}(E_{ij})\bar{v}_k$ is either 0 or has weight $\lambda_k + \alpha_{ij}$, we conclude that $\tilde{\pi}(E_{ij})\bar{v}_k = 0$ for every $k = 1, \ldots, m$. Thus, $\pi(N_{l+})\bar{v}_k = \bar{v}_k$. Since $V_0 = W$, the vector $\bar{v}_k$ cannot be fixed by $\pi(A_{l+}^o)$. Hence, $\lambda_k \neq 0$. $\qquad \square$

We now modify slightly our notations. For $s = (s_{l+1}, \ldots, s_n) \in \mathbb{R}^{n-l}$, $\sum_{i=l+1}^n s_i = 0$, denote

$$a^+(s) = \mathrm{diag}(1, \ldots, 1, e^{s_{l+1}}, \ldots, e^{s_n}).$$

For $\beta > 0$, define

$$D(\beta) = \left\{ t = (t_{ij} : \max(i,l) < j) : \sum_{\max(i,l)<j} t_{ij}^2 < \beta^2 \right\}. \tag{47}$$

Lemma 22. *Let π be a representation of G on a finite-dimensional real vector space V, V_0 be defined as in (46), and $\bar{V} = V/V_0$. Introduce a norm on $\bar{V}$. Then for every relatively compact subset $K \subseteq \bar{V}$ and $r > 0$, there exists $\alpha \in (0,1)$ and $C_0 > 0$ such that for every s such that $a^+(s) \in A_{l+}^{C_0}$ and $x \in \bar{V}$ such that $\|x\| > r$,*

$$\pi(a^+(s)n^+(D(e^{-\alpha s_{l+1}})))x \not\subseteq K. \tag{48}$$

Proof. The proof is the same as the proof of Lemma 16 in [Go02]. We will just sketch the main idea.

We need to get a lower estimate for

$$\sup\{\|\pi(a^+(s))\pi(n^+(t))x\| : t \in D(e^{-\alpha s_{l+1}})\}.$$

Let $\bar{W} = \{\bar{v} \in \bar{V} : \pi(N_{l+})\bar{v} = \bar{v}\}$, and $\mathrm{pr}_{\bar{W}} : \bar{V} \to \bar{W}$ is a projection on $\bar{W}$ that commutes with $\pi(a^+(s))$. By Lemma 21, $\bar{W}$ is spanned by weight vectors of A_{l+}^o with nonzero dominant weights. Using that the character $s \mapsto e^{s_{l+1}}$ is the smallest nontrivial dominant weight of A_{l+}^o, one concludes that for every $y \in \bar{V}$ and $a^+(s) \in A_{l+}^C$ with $C > 0$,[1]

$$\|\pi(a^+(s))y\| \gg \|\pi(a^+(s))\mathrm{pr}_{\bar{W}}(y)\| \gg e^{s_{l+1}}\|\mathrm{pr}_{\bar{W}}(y)\|. \tag{49}$$

---

[1] $A \ll B$ means $A < c \cdot B$ for some absolute constant $c > 0$.

By a lemma due to Shah [Sh96] (see [Go02, Lemma 13]),

$$\sup\{\|\mathrm{pr}_{\bar{W}}(\pi(n^+(t))x)\| : t \in D(e^{-\alpha s_{l+1}})\} \gg (e^{-\alpha s_{l+1}})^d \|x\|. \tag{50}$$

for some positive integer d. Combining (49) and (50), we get

$$\sup\{\|\pi(a^+(s))\pi(n^+(t))x\| : t \in D(e^{-\alpha s_{l+1}})\} \gg e^{cs_{l+1}} \|x\|, \tag{51}$$

where $c = 1 - \alpha d$. Choose $\alpha \in (0,1)$ such that $c > 0$. Then for $a^+(s) \in A_{l+}^C$ the right hand side of (51) gets arbitrarily large as $C \to \infty$. This proves (48). $\qquad\square$

4.2. **Proof of Theorem 20.**

Now we are ready to start the proof of Theorem 20. Let $\mathcal{X} = (\Gamma\backslash G) \cup \{\infty\}$ be the one-point compactification of $\Gamma\backslash G$. For $T > 0$, define a normalized measure on $\mathcal{X}$ by

$$\nu_T(\tilde{f}) = \frac{1}{\varrho_l(B_{l,T}^o)} \int_{B_{l,T}^o} \tilde{f}(yb^{-1}) d\varrho_l(b), \quad \tilde{f} \in C_c(\Gamma\backslash G).$$

We need to show that $\nu_T \to \nu$ as $T \to \infty$ in weak* topology. Since the space of normalized measures on $\mathcal{Z}$ is compact in weak* topology, it is enough to show that every limit point of ν_T, $T \to \infty$, is equal to ν. Let $\nu_{T_i} \to \eta$ as $T_i \to \infty$ for some normalized measure η on $\mathcal{X}$. By Lemma 18, for every $C \in \mathbb{R}$,

$$\eta(\tilde{f}) = \lim_{T_i \to \infty} \frac{1}{\varrho_l(B_{l,T_i}^o)} \int_{B_{l,T_i}^C} \tilde{f}(yb^{-1}) d\varrho_l(b) \tag{52}$$

Let $U = \{n^+(t) \in N : t_{ij} = 0 \text{ for } i < j < n\}$.

Lemma 23. *The measure η is U-invariant.*

Up to minor modifications, the proof is the same as the proof of Lemma 18 in [Go02].

Lemma 24. *For $\alpha \in (0,1)$, define*

$$\tilde{A}_{l+,T}^C = \left\{ a^+(s) \in A_{l+,T}^C : (T^2 - N(s) - l)^{1/2} > \exp\left(\max_{l<i<n}\{s_i\} - \alpha s_{l+1}\right) \right\}, \tag{53}$$

where $N(s)$ is defined in (82), and

$$\tilde{B}_{l,T}^C = \left(\tilde{A}_{l+,T}^C N_{l+} \right) \bigcap B_{l,T}^o.$$

Then for every $C > 0$,

$$\eta(\tilde{f}) = \lim_{T_i \to \infty} \frac{1}{\varrho_l(B_{l,T_i}^o)} \int_{\tilde{B}_{l,T_i}^C} \tilde{f}(yb^{-1}) d\varrho_l(b).$$

The proof is routine computation based on Lemma 28 in the Appendix. See Lemma 19 in [Go02] or the proof of Lemma 18 above for a similar argument.

Write $y = \Gamma g_0$ for some $g_0 \in G$. Let $q_s(t) = n^+(t)^{-1} a^+(s)^{-1}$.

Next, we review some deep results on distribution of polynomial trajectories due to Dani, Margulis, Shah, and Ratner. See [KSS02] and [St, §19] for more comprehensive exposition. These results will be applied to the polynomial map

$$t \mapsto g_0 q_s(t) : \mathbb{R}^m \to \mathrm{SL}(n,\mathbb{R}),$$

where $m = \frac{1}{2}(n-l)(n+l-1)$.

Let $\mathfrak{g}$ be the Lie algebra of G, and $V_G = \oplus_{i=1}^{\dim g} \wedge^i \mathfrak{g}$. Fix a norm in V_G. For every Lie subgroup H of G with Lie algebra $\mathfrak{h}$, take a unit vector $p_H \in \wedge^{\dim \mathfrak{h}} \mathfrak{h} \subseteq V_G$. Also define

$$X(H, U) = \{g \in G : gU \subseteq Hg\}.$$

Denote by $\mathcal{H}_\Gamma$ the family of all proper closed connected subgroups H of G such that $\Gamma \cap H$ is a lattice in H, and $\mathrm{Ad}(\Gamma \cap H)$ is Zariski dense in $\mathrm{Ad}(H)$.

The *singular set* of U is

$$Y = \bigcup_{H \in \mathcal{H}_\Gamma} \Gamma X(H, U) \subseteq \Gamma \backslash G.$$

The set Y is precisely the set of $y \in \Gamma \backslash G$ such that yU is not dense in $\Gamma \backslash G$.

The following facts and results will be used in the sequel:

(I) The set $\mathcal{H}_\Gamma$ is countable.

(II) For every $H \in \mathcal{H}_\Gamma$, $\Gamma \cdot p_H$ is discrete in V_G. Thus, $\Gamma N_G^1(H)$ is closed in $\Gamma \backslash G$, where
$$N_G^1(H) = \{g \in G : g \cdot p_H = p_H\}.$$

(III) Assume that Γ is not cocompact. Then there exist closed subgroups U_i, $i = 1, \ldots, r$, such that each U_i is the unipotent radical of a parabolic subgroup, ΓU_i is closed in $\Gamma \backslash G$, and for every $\varepsilon, \delta > 0$, there exists a compact set $C \subseteq \Gamma \backslash G$ such that for every bounded open convex subset $D \subseteq \mathbb{R}^m$, one of the following holds:

    1. There exist $\gamma \in \Gamma$ and $i = 1, \ldots, r$ such that

$$\sup_{t \in D} \|q_s(t)^{-1} g_0^{-1} \gamma \cdot p_{U_i}\| \leq \delta.$$

    2. $\omega(\{t \in D : \Gamma g_0 q_s(t) \notin C\}) < \varepsilon \omega(D)$, where ω is the Lebesgue measure on $\mathbb{R}^m$.

(IV) Let $\varepsilon > 0$ and $H \in \mathcal{H}_\Gamma$. For every compact set $C \subseteq \Gamma X(H, U)$, there exists a compact set $F \subseteq V_G$ such that for every neighborhood Φ of F in V_G there exists a neighborhood Ψ of C in $\Gamma \backslash G$ such that for every bounded open convex set $D \subseteq \mathbb{R}^m$, one of the following holds:

    1. There exists $\gamma \in \Gamma$ such that $q_s(D)^{-1} g_0^{-1} \gamma \cdot p_H \subseteq \Phi$.

    2. $\omega(\{t \in D : \Gamma g_0 q_s(t) \in \Psi\}) < \varepsilon \omega(D)$.

(I) is proved in [Rat91a, Theorem 1.1] and [DM93, Theorem 2.1]. For the proof of (II), see [DM93, Theorem 3.4]. (III) is a special case of [Sh96, Theorems 2.1–2.2]. (IV) is based on [Sh94, Proposition 5.4]. It is formulated in [Sh96].

To simplify our notations, we put $V = V_G$. Let V_0 be defined as in (46).

Lemma 25. *For $H \in \mathcal{H}_\Gamma$, $g_0^{-1} \Gamma \cdot p_H \subseteq V - V_0$.*

Proof. Suppose that $g_0^{-1} \gamma \cdot p_H \in V_0$ for some $\gamma \in \Gamma$. Then $(\gamma^{-1} g_0 B_l^o g_0^{-1} \gamma) \cdot p_H = p_H$. Thus, $\gamma^{-1} g_0 B_l^o g_0^{-1} \gamma \subseteq N_G^1(H)$. By (II), $\Gamma N_G^1(H)$ is not dense in $\Gamma \backslash G$. It follows that $\Gamma g_0 B_l^o$ is not dense in $\Gamma \backslash G$ too. This is a contradiction. $\qquad\square$

In the case when Γ is not cocompact, we prove the following lemma:

Lemma 26. $\eta(\{\infty\}) = 0$.

Proof. Write

$$V = V_0 \oplus V_1, \tag{54}$$

where V_1 is A_{l+}^o-invariant complement. For a vector $v \in V$, denote by $v_0 \in V_0$ and $v_1 \in V_1$ its components with respect to the decomposition (54). Fix norms on V_0 and V_1. Define a norm on V by

$$\|v\| = \max\{\|v_0\|, \|v_1\|\}, \quad v_0 \in V_0, \ v_1 \in V_1.$$

The space V_1 is naturally isomorphic with V/V_0. The norm on V_1 induces a norm on V/V_0 through this isomorphism.

We use (III). Let $\varepsilon, \delta > 0$. Let

$$P = \bigcup_{i=1}^{r} g_0^{-1} \Gamma \cdot p_{U_i},$$

and $P_1 = \{p \in P : \|p_0\| > \delta\}$. For $p \in P_1$,

$$\|q_s(0)^{-1} p\| \geq \|p_0\| > \delta. \tag{55}$$

By Lemma 25, $P \subseteq V - V_0$, and by (II), P is discrete. Therefore, there exists $r > 0$ such that $\|p_1\| > r$ for $p \in P - P_1$. Since the factor-map $V \to V/V_0$ is continuous and B_l^o-equivariant, for some $M > 0$,

$$\|q_s(t)^{-1} \cdot v\| \geq M \|q_s(t)^{-1} \cdot \bar{v}\|, \quad v \in V. \tag{56}$$

Now we apply Lemma 22 with $K = \{\bar{v} \in V/V_0 : \|\bar{v}\| \leq \frac{\delta}{M}\}$. There exist $\alpha \in (0, 1)$ and $C_0 > 0$ such that for every s such that $a^+(s) \in A_{l+}^{C_0}$ and every $\bar{v} \in V/V_0$ such that $\|\bar{v}\| > r$,

$$q_s(D(e^{-\alpha s_{l+1}}))^{-1} \cdot \bar{v} \not\subseteq K.$$

In particular, this holds for $\bar{v} = \bar{p}$ with $p \in P - P_1$. Thus, by (56),

$$\sup_{t \in D(e^{-\alpha s_{l+1}})} \|q_s(t)^{-1} \cdot p\| > \delta \tag{57}$$

for $p \in P - P_1$. In fact, (57) holds for $p \in P_1$ because of (55). Thus, the case (a) of (III) does not occur when $a^+(s) \in A_{l+,T_i}^{C_0}$ and D is a bounded open convex subset such that $D \supseteq D(e^{-\alpha s_{l+1}})$. It follows that for some compact set $C \subseteq \Gamma \backslash G$,

$$\omega(\{t \in D : \Gamma g_0 q_s(t) \notin C\}) < \varepsilon \omega(D). \tag{58}$$

when $a^+(s) \in A_{l+,T_i}^{C_0}$ and $D \supseteq D(e^{-\alpha s_{l+1}})$.

We have

$$\eta(\tilde{f}) = \lim_{T_i \to \infty} \frac{1}{\varrho_l(B_{l,T_i}^o)} \int_{\tilde{A}_{l+,T_i}^C} \int_{D_{s,T_i}} \tilde{f}(\Gamma g_0 q_s(t)) \delta_l^+(s) dt^+ ds^+, \quad \tilde{f} \in C_c(\Gamma \backslash G), \tag{59}$$

where

$$D_{s,T_i} = \left\{ n(t) \in N : \sum_{i \leq l; l < j} t_{ij}^2 + \sum_{l < i < j} e^{2s_i} t_{ij}^2 < T_i^2 - N(s) - l \right\}, \tag{60}$$

and $N(s)$ is defined in (82). Note that D_{s,T_i} contains $D(\beta)$, which is defined in (47), for

$$\beta < (T_i^2 - N(s) - l)^{1/2} \exp\left(-\max_{l+1 \leq i \leq n-1} \{s_i\}\right).$$

When $a^+(s) \in \tilde{A}^{C_0}_{l+,T_i}$, the right hand side is greater then $e^{-\alpha s_{l+1}}$ (see (53)). Therefore, $D_{s,T_i} \supseteq D(e^{-\alpha s_{l+1}})$ when $a^+(s) \in \tilde{A}^{C_0}_{l+,T_i}$. By (58),

$$\omega\left(\{t \in D_{s,T_i} : \Gamma g_0 q_s(t) \notin C\}\right) < \varepsilon \omega(D_{s,T_i}) \quad \text{for } a^+(s) \in \tilde{A}^{C_0}_{l+,T_i}. \tag{61}$$

Let χ_C be the characteristic function of the set C. Take $\tilde{f} \in C_c(\Gamma \backslash G)$ such that $\chi_C \leq \tilde{f} \leq 1$. Then using (59) and (61), we get

$$\begin{aligned}
\eta(\mathrm{supp}(\tilde{f})) &\geq \lim_{T_i \to \infty} \frac{1}{\varrho_l(B^o_{l,T_i})} \int_{\tilde{A}^{C_0}_{l+,T_i}} \int_{D_{s,T_i}} \chi_C(\Gamma g_0 q_s(t)) \delta^+_l(s) dt^+ ds^+ \\
&\geq \lim_{T_i \to \infty} \frac{1}{\varrho_l(B^o_{l,T_i})} \int_{\tilde{A}^{C_0}_{l+,T_i}} (1-\varepsilon) \omega(D_{s,T_i}) \delta^+_l(s) ds^+ \\
&= (1-\varepsilon) \lim_{T_i \to \infty} \frac{\varrho_l(\tilde{B}^{C_0}_{l,T_i})}{\varrho_l(B^o_{l,T_i})} = 1 - \varepsilon.
\end{aligned}$$

Hence, $\eta(\{\infty\}) \leq \eta(\mathrm{supp}(\tilde{f})^c) \leq \varepsilon$ for every $\varepsilon > 0$. $\qquad\square$

Lemma 27. $\eta(Y) = 0$.

Proof. Since $\mathcal{H}_\Gamma$ is countable, it is enough to show that $\eta(\Gamma X(H,U)) = 0$ for every $H \in \mathcal{H}_\Gamma$. Moreover, it is enough to show that $\eta(C) = 0$ for every compact set $C \subseteq \Gamma X(H,U)$.

We use the notations from the proof of Lemma 26, in particular, decomposition (54).

We apply (IV). Take $\varepsilon > 0$. Let F be a compact subset of V as in (IV). Take a relatively compact neighborhood Φ of F. Let $\Psi \supset C$ be as in (IV). Denote $P = g_0^{-1}\Gamma \cdot p_H$ and $P_1 = \{p \in P : \|p_0\| > \delta\}$ with $\delta = \sup\{\|v_0\| : v \in \Phi\}$. Then (55) holds, so that

$$q_s(0) \cdot p \notin \Phi. \tag{62}$$

As in the proof of the previous lemma, there exists $r > 0$ such that $\|p_1\| > r$ for $p \in P - P_1$, and applying Lemma 22, we deduce that there exist $\alpha \in (0,1)$ and $C_0 > 0$ such that for every s such that $a^+(s) \in A^{C_0}_{l+}$ and every $p \in P - P_1$,

$$q_s(D(e^{-\alpha s_{l+1}}))^{-1} \cdot p \not\subseteq \Phi. \tag{63}$$

By (62), (63) holds for every $p \in P$. Thus, case (a) of (IV) fails. Therefore, case (b) holds:

$$\omega(\{t \in D : \Gamma g_0 q_s(t) \in \Psi\}) < \varepsilon \omega(D), \tag{64}$$

when $a^+(s) \in A^{C_0}_{l+,T_i}$ and D is an open convex set such that $D \supseteq D(e^{-\alpha s_{l+1}})$. Recall that D_{s,T_i} was defined in (60). It is easy to see from (60) that $D_{s,T_i} \supseteq D(e^{-\alpha s_{l+1}})$ when $a^+(s) \in \tilde{A}^{C_0}_{l+,T_i}$. Thus, (64) holds for $D = D_{s,T_i}$ with $a^+(s) \in \tilde{A}^{C_0}_{l+,T_i}$.

Take a function $\tilde{f} \in C_c(\Gamma \backslash G)$ such that $\tilde{f} = 1$ on C, $\mathrm{supp}(\tilde{f}) \subseteq \Psi$, and $0 \leq \tilde{f} \leq 1$. Let χ_Ψ be the characteristic function of Ψ. Then using (59) and (64) with $D = D_{s,T_i}$, we get

$$\eta(C) \leq \lim_{T_i \to \infty} \frac{1}{\varrho_l(B^o_{l,T_i})} \int_{\tilde{A}^{C_0}_{l+,T_i}} \int_{D_{s,T_i}} \chi_\Psi(\Gamma g_0 q_s(t)) \delta^+_l(s) dt^+ ds^+$$

$$\leq \lim_{T_i \to \infty} \frac{1}{\varrho_l(B^o_{T_i})} \int_{\tilde{A}^{C_0}_{l+,T_i}} \varepsilon \omega(D_{s,T_i}) \delta^+_l(s) ds^+ = \varepsilon \lim_{T_i \to \infty} \frac{\varrho_l(\tilde{B}^{C_0}_{l,T_i})}{\varrho_l(B^o_{T_i})} = \varepsilon.$$

This shows that $\eta(C) = 0$. Hence, $\eta(Y) = 0$. $\qquad\square$

By Lemma 23, the measure η is U-invariant. Consider the ergodic decomposition of η into U-ergodic measures. By Ratner's measure classification [Rat91a], an ergodic component of η is either G-invariant or supported on $Y \cup \{\infty\}$. By Lemmas 26 and 27, the set of ergodic components of the second type has measure 0. Therefore, η is G-invariant, and $\eta = \nu$. This proves Theorem 20.

5. UNIFORM DISTRIBUTION FOR A LATTICE IN $\mathrm{Sp}(n, \mathbb{R})$

5.1. **Density of orbits.**

Proof of Theorem 5. Clearly, the condition is neccesary for denseness. Suppose that the condition is satisfied. Let $\{e_i : i = 1, \ldots, 2n\}$ be the standard basis of $\mathbb{R}^{2n}$, and $e = (e_1, \ldots, e_n)$. Then e is an isotropic frame, and by Witt's Theorem, the space of isotropic n-frames is Ge. The stabilizer of e in G is

$$U_0 = \left\{ \left(\begin{array}{c|c} E & X \\ \hline 0 & E \end{array} \right) : {}^tX = X \right\}.$$

Let $g_0 \in G$ be such that $g_0 v = e$. Then the stabilizer of v in G is $U = g_0^{-1} U_0 g_0$. It is not hard to check that any U_0-invariant subspace is either contained in $\langle e_1, \ldots, e_n \rangle$ or contains it. It follows that any U-invariant subspace is either contained in $V_0 \overset{def}{=} \langle v_1, \ldots, v_n \rangle$ or contains it.

As in the proof of Theorem 1, $\overline{\Gamma U} = \Gamma H$ where H is the connected component of the smallest real $\mathbb{Q}$-algebraic subgroup containing U, and the radical of H is unipotent. Let R be the radical of H. The space of R-fixed vectors V^R is defined over $\mathbb{Q}$ and H-invariant. Since R is unipotent, $V^R \neq 0$. Thus, by the condition on v, $V^R \not\subseteq V_0$. Then $V_0 \subseteq V^R$. Suppose that $V^R \neq \mathbb{R}^{2n}$. Since $V_0^\perp = V_0$, $J|_{V^R}$ is degenerate. Then $0 \neq \mathrm{Rad}(J|_{V^R}) \subseteq V_0$. This is a contradiction because the space $\mathrm{Rad}(J|_{V^R})$ is defined over $\mathbb{Q}$. Hence, $V^R = \mathbb{R}^{2n}$, $R = 1$, and H is semisimple.

We claim that $H = G$. Let $H_0 = g_0 H g_0^{-1}$. Denote by $\mathfrak{g}$, $\mathfrak{h}$, and $\mathfrak{u}$ the Lie algebras of G, H_0, and U_0 respectively. The Killing form on $\mathfrak{g}$ is defined by $k(x, y) = \mathrm{Tr}(xy)$ for $x, y \in \mathfrak{g}$. Since H_0 is semisimple, k is nondegenerate on $\mathfrak{h}$.

Recall the root decomposition for $\mathfrak{g}$. A Cartan subalgebra of $\mathfrak{g}$ is

$$\mathfrak{a} = \{\mathrm{diag}(h_1, \ldots, h_n, -h_1, \ldots, -h_n) : h_i \in \mathbb{R}\}.$$

Let $\alpha_i(h) = h_i$ for $h \in \mathfrak{a}$. The root system of $\mathfrak{g}$ is

$$\Delta = \{\alpha_i - \alpha_j, \pm(\alpha_p + \alpha_q) : 1 \leq i \neq j \leq n, 1 \leq p \leq q \leq n\},$$

and the following root space decomposition holds:

$$\begin{aligned}
\mathfrak{g}_{\alpha_i - \alpha_j} &= \langle E_{i,j} - E_{j+n, i+n} \rangle, \\
\mathfrak{g}_{\alpha_p + \alpha_q} &= \langle E_{p, q+n} + E_{q, p+n} \rangle, \\
\mathfrak{g}_{-\alpha_p - \alpha_q} &= \langle E_{p+n, q} + E_{q+n, p} \rangle, \\
\mathfrak{g} &= \mathfrak{a} \oplus \sum_{\alpha \in \Delta} \mathfrak{g}_\alpha.
\end{aligned} \tag{65}$$

Since $U_0 \subseteq H_0$,

$$\mathfrak{u} = \sum_{1 \leq p \leq q \leq n} \mathfrak{g}_{\alpha_p + \alpha_q} \subseteq \mathfrak{h}.$$

Note that $k(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0$ if $\alpha + \beta \neq 0$. Since the Killing form k is nondegenerate on $\mathfrak{h}$, the projection map from $\mathfrak{h}$ to the space $\sum_{1 \leq p \leq q \leq n} \mathfrak{g}_{-\alpha_p - \alpha_q}$ with respect to the decomposition (65) is surjective. Thus for $1 \leq p \leq q \leq n$, there exists $h_{pq} = x_{pq} + \tilde{h}_{pq} \in \mathfrak{h}$ where x_{pq} is a generator of the space $\mathfrak{g}_{-\alpha_p - \alpha_q}$, and $\tilde{h}_{pq}$ is in the normalizer of $\mathfrak{u}$. Then

$$\mathfrak{h} \supseteq [h_{pq}, \mathfrak{u}] + \mathfrak{u} \supseteq [x_{pq}, u], \quad 1 \leq p \leq q \leq n.$$

Using that $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$ for $\alpha, \beta, \alpha + \beta \in \Delta$, we conclude that

$$\sum_{\alpha \in \Delta} \mathfrak{g}_\alpha \subseteq \mathfrak{h}.$$

It follows that $\mathfrak{h} = \mathfrak{g}$, and $H = G$. Finally,

$$\overline{\Gamma v} = \overline{\Gamma U v} \supseteq \overline{\Gamma U} v = G v = \mathcal{W}_n.$$

$\square$

5.2. Iwasawa decomposition for $\mathbf{Sp}(n, \mathbb{R})$.

Let $G = \mathrm{Sp}(n, \mathbb{R})$. We use the following notations:

$$
\begin{aligned}
K &= G \cap \mathrm{SO}(2n, \mathbb{R}), \\
N_+ &= \left\{ \begin{pmatrix} E & N \\ 0 & E \end{pmatrix} : {}^t N = N \right\}, \\
N_- &= \left\{ \begin{pmatrix} M & 0 \\ 0 & {}^t M^{-1} \end{pmatrix} : M \text{ is upper triangular, unipotent} \right\}, \\
A &= \left\{ \begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix} : A \text{ is positive, diagonal} \right\}, \\
B &= N_- A, \\
N &= N_- N_+.
\end{aligned}
$$

We have Iwasawa decomposition:

$$(k, n, a) \mapsto kna : K \times N \times A \to G$$

(see, for example, [Te, p. 286]). This map is a diffeomorphism. It is easy to check that the product map $N_- \times N_+ \to N$ is a diffeomorphism, N_- normalizes N_+, and A normalizes N_+. Thus, modified Iwasawa decomposition holds:

$$(k, b, n) \mapsto kbn : K \times B \times N_+ \to G. \tag{66}$$

Fix $g_0 \in G$. We also have decomposition:

$$(k, b, n) \mapsto kbg_0 n^{g_0} : K \times B \times N_+ \to G. \tag{67}$$

For $g \in G$, define $k_g \in K$, $b_g \in B$, and $n_g \in N_+$ such that

$$g = k_g b_g g_0 n_g^{g_0}.$$

Also define $b'_g \in \mathrm{GL}(n, \mathbb{R})$ such that

$$b_g = \begin{pmatrix} b'_g & 0 \\ 0 & {}^t(b'_g)^{-1} \end{pmatrix}.$$

Let μ be a Haar measure on G, and $\bar{\mu}$ be the measure on $\Gamma \backslash G$ such that

$$\int_G f d\mu = \int_{\Gamma \backslash G} \sum_{\gamma \in \Gamma} f(\gamma g) d\bar{\mu}(g), \quad f \in C_c(G).$$

Let ϱ be the Lebesgue measure on $N_+ \simeq \mathbb{R}^{\frac{n(n+1)}{2}}$, and ν be the measure on $G/N_+^{g_0}$ such that

$$\int_G f d\mu = \int_{G/N_+^{g_0}} \int_{N_+} f(gn^{g_0}) d\varrho(n) d\nu(g), \quad f \in C_c(G).$$

Note that

$$\varrho(N_{+,T}) \sim CT^{\frac{n(n+1)}{2}} \quad \text{as} \quad T \to \infty \tag{68}$$

for some $C > 0$.

5.3. **Uniform distribution.**

Proof of Theorem 6. We can write $v^0 = g_0^{-1} e^0$ for some $g_0 \in G$, where $e^0 = (e_1, \ldots, e_n)$ is the standard frame. Without loss of generality, $g_0^{-1} = k_0 b_0$ for some $k_0 \in K$, and $b_0 \in B$ (see (66)).

Let $Y = KB g_0$. By (67), the product map $Y \times N_+^{g_0} \to G$ is a diffeomorphism. For $g \in G$, denote $y_g = k_g b_g g_0$, the Y-component of g. The map

$$\alpha : Y \to \mathcal{W}_n \simeq G/N_+^{g_0} : y \mapsto yv^0$$

is a diffeomorphism. Denote by ν_1 the measure on Y which is the pull-back of the measure ν by the map α.

For $g \in G$, $gv^0 = y_g v^0$. This shows that $gv^0 \in \Omega$ iff $y_g \in \alpha^{-1}(\Omega)$. Write

$$n_g = \begin{pmatrix} E & l_g \\ 0 & E \end{pmatrix}.$$

Then

$$\|g\| = \|k_g b_g n_g g_0\| = \|b_g n_g b_0^{-1}\| = \left\| \begin{pmatrix} * & b'_g l_g ({}^t b'_0) \\ 0 & * \end{pmatrix} \right\|.$$

Thus,

$$\|g\|^2 = \|\hat{c}_g(n_g d_g)\|^2 + e_g,$$

where

$$c_g = b_g, \quad d_g = \begin{pmatrix} (b'_0)^{-1} b'_g & 0 \\ 0 & ({}^t b'_0)({}^t b'_g)^{-1} \end{pmatrix},$$

and e_g is a continuous function depending only on the B-components of g. We can use Proposition 10 with $H = N_+$, $h_g = n_g$, and $m_g = 1$. Since $\Gamma \cdot v^0$ is dense in $\mathcal{W}_n$, $\Gamma N_+^{g_0}$ is dense in G. By (68), the condition (5) holds for $H = N_+$. Since N_+ is unipotent, the condition (6) for $H = N_+$ holds too [Sh94].

Applying Proposition 10, we get

$$N_T(\Omega, v^0) \sim \left(\frac{1}{\bar{\mu}(\Gamma \backslash G)} \int_{\alpha^{-1}(\Omega)} \frac{d\nu_1(y)}{\Delta_H(c_y)} \right) \varrho(N_{+,T})$$

as $T \to \infty$, where Δ_H is defined in (7). Thus, by (68),

$$N_T(\Omega, v^0) \sim \lambda_{v^0}(\Omega) T^{\frac{n(n+1)}{2}} \quad \text{as} \quad T \to \infty,$$

where

$$\lambda_{v^0}(\Omega) = \frac{C}{\bar{\mu}(\Gamma \backslash G)} \int_\Omega \frac{d\nu(x)}{\Delta_{N_+}(c_x)}.$$

$\square$

APPENDIX

Proof of Lemma 16 (part 2). To find the constant $d_{n,l}$, we calculate measures of the set

$$D = \{v = (v_1, \ldots, v_l) \in \mathcal{V}_{n,l} : \|v_i\| < 1, 1 \le i \le l\}.$$

Denote by V_k the Lebesgue measure of a k-dimensional unit ball. Recall that

$$V_k = \frac{\pi^{k/2}}{\Gamma(1 + k/2)}. \tag{69}$$

Clearly,

$$\int_D dv = V_n^l = \frac{\pi^{nl/2}}{\Gamma(1 + n/2)^l}. \tag{70}$$

For $k \in K$, $n \in N_{l-}$, and $a \in A_{l-}^o$, $knae^0 \in D$ iff $\|knae_i\| < 1$ for $i = 1, \ldots, l$. We have

$$\|kn^-(t)a(s^-)e_i\|^2 = \exp(2s_i^-)\left(1 + t_{1i}^2 + \cdots + t_{i-1i}^2\right). \tag{71}$$

Let as introduce new coordinates on A_{l-}^o: $a_i = \exp(s_i^-)$, $1 \le i \le l$. The Haar measure on A_{l-}^o (23) is given by $da = \prod_{i=1}^l \frac{da_i}{a_i}$. By (71), the set of $(k, n^-(t), a) \in K \times N_{l-} \times A_{l-}^o$ such that $kn^-(t)ae^0 \in D$ is described by conditions:

$$0 < a_i < 1 \qquad i = 1, \ldots, l,$$
$$\|t_{*i}\| < \left(\frac{1 - a_i^2}{a_i^2}\right)^{1/2} \qquad i = 2, \ldots, l.$$

Thus,

$$\int\limits_{kn^- a^- e^0 \in D} \delta_l^-(a^-)^n dk dn^- da^-$$

$$= \int_0^1 \cdots \int_0^1 \prod_{i=2}^l V_{i-1} \left(\frac{1-a_i^2}{a_i^2}\right)^{\frac{i-1}{2}} \delta_l^-(a^-)^n da^-$$

$$= \prod_{i=1}^l V_{i-1} \int_0^1 (1-a_i^2)^{\frac{i-1}{2}} a_i^{n-i} da_i$$

$$= \prod_{i=1}^l \frac{V_{i-1}}{2} \int_0^1 (1-b_i)^{\frac{i-1}{2}} b_i^{\frac{n-i-1}{2}} db_i$$

$$= 2^{-l} \prod_{i=1}^l V_{i-1} B\left(\frac{i+1}{2}, \frac{n-i+1}{2}\right)$$

$$= \frac{\pi^{l(l-1)/4}}{2^l \Gamma(1+n/2)^l} \prod_{i=1}^l \Gamma\left(\frac{n-i+1}{2}\right). \tag{72}$$

In the last step, we have used (69) and the well-known identity for Γ-function and B-function. Finally, by (70) and (72),

$$d_{n,l} = \frac{2^l \pi^{l(2n-l+1)/4}}{\Gamma(n/2)\Gamma((n-1)/2)\cdots\Gamma((n-l+1)/2)}. \tag{73}$$

$\square$

Proof of Lemma 17. Let

$$d(\bar{a}) = \mathrm{diag}\,(1, \ldots, 1, a_{l+1}, \ldots, a_n)$$

for $\bar{a} = (a_{l+1}, \ldots, a_n) \in \mathbb{R}^{n-l}$, and

$$\alpha(\lambda) = \mathrm{diag}\left(1, \ldots, 1, \lambda^{\frac{1}{n-l}}, \ldots, \lambda^{\frac{1}{n-l}}\right).$$

For $b = n^+(t)d(\bar{a})$, define

$$\Lambda(b) = \sum_{l < i \le n} a_i^2 + \sum_{\max(i,l) < j} a_j^2 t_{ij}^2. \tag{74}$$

Note that $\Lambda(b) = \|b\|^2 - l$. Thus, it is enough to compute asymptotics of the function

$$\phi(x) \stackrel{def}{=} \int_{\Lambda(b) < x} d\varrho_l(b).$$

as $x \to \infty$. By Tauberian theorem [Wid, Ch. V, Theorem 4.3], it can be deduced from asymptotics of the function

$$\psi(\lambda) \stackrel{def}{=} \int_0^\infty e^{-\lambda x} d\phi(x) = \int_{B_l^o} \exp\{-\lambda\Lambda(b)\} d\varrho_l(b). \tag{75}$$

as $\lambda \to 0+$. It is more convenient to work with the function

$$\tilde{\psi}(\lambda) = \psi\left(\lambda^{\frac{2}{n-l}}\right). \tag{76}$$

Let $\bar{B}_l^o = N_{l+}d(\mathbb{R}_+^d) = B_l^o\alpha(\mathbb{R}_+)$. One can check that

$$\int_{N_{l+}\times\mathbb{R}_+^{n-l}} f(nd(\bar{a}))dn^+ \frac{da_{l+1}}{a_{l+1}} \cdots \frac{da_n}{a_n} = \int_{B_l^o\times\mathbb{R}_+} f(b\alpha(\lambda))d\varrho_l(b)\frac{d\lambda}{\lambda} \tag{77}$$

for $f \in L^1(\bar{B}_l^o)$. (In fact, each of the integral defines a right Haar measure on $\bar{B}_l^o$.)

Consider Mellin transform of the function $\tilde{\psi}$:

$$
\begin{aligned}
F(z) &= \int_0^\infty \tilde{\psi}(\lambda)\lambda^z \frac{d\lambda}{\lambda} \overset{(75)}{=} \int_{B_l^o\times\mathbb{R}_+} \exp\left\{-\Lambda(b\alpha(\lambda))\right\}\lambda^z d\varrho_l(b)\frac{d\lambda}{\lambda} \\
&\overset{(77)}{=} \int_{N_{l+}\times\mathbb{R}_+^{n-l}} \exp\left\{-\Lambda(n^+d(\bar{a}))\right\}\left(\prod_{i=l+1}^n a_i\right)^z dn^+ \frac{da_{l+1}}{a_{l+1}}\cdots\frac{da_n}{a_n} \\
&\overset{(74)}{=} \int_{N_{l+}\times\mathbb{R}_+^{n-l}} \exp\left\{-\sum_{\max(i,l)<j}(a_jt_{ij})^2 - \sum_{i=l+1}^n a_i^2\right\} \\
&\qquad \times \left(\prod_{i=l+1}^n a_i\right)^{z-1} \prod_{\max(i,l)<j} dt_{ij} \prod_{i=l+1}^n da_i.
\end{aligned}
$$

Using that $\int_\mathbb{R} e^{-x^2}dx = \sqrt{\pi}$, we get

$$F(z) = \pi^{\frac{(n+l-1)(n-l)}{4}} \int_{\mathbb{R}_+^{n-l}} \exp\left\{-\sum_{i=l+1}^n a_i^2\right\}\left(\prod_{i=l+1}^n a_i^{z-i}\right) da_{l+1}\ldots da_n.$$

Making substitution $u_i = a_i^2$, we get

$$F(z) = \frac{\pi^{\frac{(n+l-1)(n-l)}{4}}}{2^{n-l}} \prod_{j=l+1}^n \Gamma\left(\frac{z-j+1}{2}\right). \tag{78}$$

By Mellin inversion formula, for sufficiently large u,

$$\tilde{\psi}(\lambda) = \frac{1}{2\pi i}\int_{\mathrm{Re}(z)=u} F(z)\lambda^{-z}dz. \tag{79}$$

Since Γ-function decays fast on vertical strips, we can shift the line of integration to the left. By (78), the first pole of $F(z)$ occurs at $z = n-1$. Therefore, it follows from (79) that

$$\tilde{\psi}(\lambda) \sim \frac{\pi^{\frac{(n+l-1)(n-l)}{4}}}{2^{n-l-1}} \prod_{j=l+1}^{n-1} \Gamma\left(\frac{n-j}{2}\right)\lambda^{-(n-1)} \ \text{ as } \ \lambda \to 0+.$$

By (76),

$$\psi(\lambda) \sim \frac{\pi^{\frac{(n+l-1)(n-l)}{4}}}{2^{n-l-1}} \prod_{j=l+1}^{n-1} \Gamma\left(\frac{n-j}{2}\right)\lambda^{-\frac{(n-1)(n-l)}{2}} \ \text{ as } \ \lambda \to 0+.$$

Finally, the asymptotic estimate for $\phi(x)$ as $x \to \infty$ follows from Tauberian Theorem [Wid, Ch. V, Theorem 4.3]. We have

$$\gamma_{n,l} = \frac{\pi^{(n+l-1)(n-l)/4}}{2^{n-l-1}\Gamma\left(\frac{(n-1)(n-l)}{2}+1\right)} \prod_{j=l+1}^{n-1} \Gamma\left(\frac{n-j}{2}\right). \tag{80}$$

This proves the lemma. $\qquad\square$

The following lemma is used in the proof of Lemma 18.

Lemma 28. *For $C \in \mathbb{R}$ and $T > 0$,*

$$\varrho_l(B_{l,T}^C) = c_{n,l} \int_{A_T^C} \left(T^2 - N(s^+) - l\right)^{\frac{(n-l)(n+l-1)}{4}} \exp\left\{\sum_{k=l+1}^{n} (n-k)s_k^+\right\} ds^+, \tag{81}$$

where

$$c_{n,l} = \frac{\pi^{(n-l)(n+l-1)/4}}{\Gamma(1+(n-l)(n+l-1)/4)},$$

$$N(s^+) = \sum_{i=l+1}^{n} \exp\{2s_i^+\}. \tag{82}$$

Proof. Note that

$$B_{l,T}^C = \left\{a(s^+)n(t) : l + N(s^+) + \sum_{l<i<j\le n} \exp\{2s_i^+\}t_{ij}^2 + \sum_{1\le i\le l;l<j\le n} t_{ij}^2 < T^2\right\}.$$

We use the formula (24) for ϱ_l and make the change of variables

$$t_{ij} \to \exp\{-s_i^+\}t_{ij}$$

for $l < i < j \le n$. The formula (81) follows from the fact that the volume of a unit ball in $\mathbb{R}^m$ is $\pi^{m/2}/\Gamma(1+m/2)$. $\qquad\square$

Proof of Lemma 18. For $i_0 = l+1, \ldots, n-1$, put

$$A_{l+,T}^{i_0} = \{a(s^+) \in A_{l+,T}^o : s_{i_0}^+ \le C\} \quad \text{and} \quad B_{l,T}^{i_0} = \{a(s^+)n(t) \in B_{l,T}^o : s_{i_0}^+ \le C\}.$$

We claim that $\varrho_l(B_{l,T}^{i_0}) = o(\varrho(B_{l,T}^o))$ as $T \to \infty$. If $a(s^+) \in A_{l+,T}^o$, then $s_i^+ < \log T$ for every $i = l+1, \ldots, n$. Then as in Lemma 28,

$$\varrho_l(B_{l,T}^{i_0}) \le c_{n,l}T^{\frac{(n-l)(n+l-1)}{2}} \int_{A_{l+,T}^{i_0}} \exp\left\{\sum_{k=l+1}^{n} (n-k)s_k^+\right\} ds^+$$

$$\ll T^{\frac{(n-l)(n+l-1)}{2}} \prod_{l<k<n,k\ne i_0} \int_{-\infty}^{\log T} \exp\{(n-k)s_k^+\}ds_k^+$$

$$\ll T^{(n-1)(n-l)-(n-i_0)}.$$

Now the claim follows from (36). Since

$$B_{l,T}^o - B_{l,T}^C = \bigcup_{l<i_0<n} B_{l,T}^{i_0},$$

we have

$$\varrho_l(B^o_{l,T} - B^C_{l,T}) = o(\varrho_l(B^o_{l,T})) \ \text{ as } \ T \to \infty.$$

Therefore, $\varrho_l(B^C_{l,T}) \sim \varrho_l(B^o_{l,T})$ as $T \to \infty$.   □

Proof of Corollary 4. By Theorem 1, Γv^0 is dense in $\mathcal{V}_{n,l}$. By Theorem 3, (44) holds. The volume of $\Gamma \backslash G$ was computed by Minkowski. For the measure $\bar{\mu}$, we have

$$\bar{\mu}(\Gamma \backslash G) = 2^{-(n-1)} \prod_{i=2}^{n} \pi^{-i/2} \Gamma(i/2) \zeta(i)$$

(see [Sh00, Theorem 5.6]).

Therefore,

$$N_T(\Omega, v^0) \sim b_{n,l} \mathrm{Vol}(v^0)^{1-n} \left(\int_\Omega \frac{dv}{\mathrm{Vol}(v)} \right) T^{(n-1)(n-l)} \quad \text{as} \quad T \to \infty,$$

where

$$b_{n,l} = \frac{a_{n,l}}{\bar{\mu}(\Gamma \backslash G)} = \frac{\pi^{n(n-l)/2}}{\Gamma\left(\frac{(n-1)(n-l)}{2} + 1\right) \Gamma\left(\frac{n-l}{2}\right)} \prod_{i=2}^{n} \zeta(i)^{-1}. \tag{83}$$

Here we used that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.   □

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Department of Mathematics, University of Michigan, Ann Arbor, MI 48109

E-mail address: gorodnik@umich.edu